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AUTHOR Beatty, Leslie; And Others

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This is part two of a two-part SMSG elementary school text for sixth-grade students. The content is aimed at the development of some of the fundamental concepts of mathematics such as number, numeration, the operations of arithmetic, and intuitive geometry. Chapter topics include division of rational numbers, volume, organizing and describing data, and sets and circles. (MP)

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STUDENT'S TEXT

UNIT NO. 34

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MATHEMATICS FOR THE ELEMENTARY SCHOOL

GRADE 6

PART II

SE 027 912

SCHOOL MATHEMATICS STUDY GROUP

YALE UNIVERSITY PRESS

Mathematics for the Elementary School, Grade 6

Student's Text, Part II

REVISED EDITION

Prepared under the supervision of the
Panel on Elementary School Mathematics
of the School Mathematics Study Group:

Leslie Beatty	Chula Vista City School District, Chula Vista, California
E. Glenadine Gibb	Iowa State Teachers College, Cedar Falls, Iowa
W. T. Guy	University of Texas
S. B. Jackson	University of Maryland
Irene Sauble	Detroit Public Schools
M. H. Stone	University of Chicago
J. F. Weaver	Boston University
R. L. Wilder	University of Michigan

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Chapter 6

DIVISION OF RATIONAL NUMBERS

SOME FACTS YOU KNOW ABOUT RATIONAL NUMBERS

The whole numbers are useful in answering questions about how many objects are in a set of objects; but to answer questions about the parts of a single object, or a part of a set of objects, numbers of a new kind are used. The new set of numbers is called the set of rational numbers.

Numbers named by numerals like 0, 8, 0.9, $\frac{7}{3}$, $4\frac{2}{5}$, and $\frac{1}{6}$ are rational numbers of arithmetic.

Some rational numbers such as 0, 14, 198, and 7033 are also whole numbers.

A fraction is one kind of symbol, or name, for a rational number. Fractions we have used in our work so far are written with two numerals separated by a bar. The number named by the numeral below the bar is called the denominator and must be a counting number. The number named above the bar is called the numerator and must be a whole number.

Every rational number has many fraction names. Whole numbers may be named by fractions whose denominators are 1.

$$0, \ 1, \ 2, \ 3, \ \ldots$$

$$\frac{0}{1}, \ \frac{1}{1}, \ \frac{2}{1}, \ \frac{3}{1}, \ \ldots$$

Whole numbers also may be named by fractions having denominators other than one.

$$\frac{12}{6} = 2,\ \frac{24}{3} = 8,\ \frac{8}{2} = 4,\ \frac{16}{4} = 4$$

Notice that whenever a whole number is named by a fraction, the numerator is always a multiple of the denominator.

Rational numbers like $\frac{2}{3}$, $\frac{4}{5}$, and $\frac{1}{8}$ also have many fraction names. Any rational number named by a fraction may be renamed by multiplying both the numerator and denominator by any counting number greater than one. Why do we say, "greater than 1"?

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15} \ \ldots$$

$$\frac{4}{5} = \frac{8}{10} = \frac{12}{15} = \frac{16}{20} = \frac{20}{25} \ \ldots$$

$$\frac{7}{8} = \frac{14}{16} = \frac{21}{24} = \frac{28}{32} = \frac{35}{40} \ \ldots$$

Can you think of this way to rename a rational number named by a fraction in a different way? Could we think of it as multiplying the number by 1, and naming 1 by a fraction with denominator greater than 1?

$$\frac{1}{2} \times \frac{2}{2} = \frac{2}{4}$$

$$\frac{1}{2} \times \frac{3}{3} = \frac{3}{6}$$

$$\frac{1}{2} \times \frac{4}{4} = \frac{4}{8}$$

The simplest fraction name for a rational number is the one in which the numerator and denominator have no common factor except 1.

$$\frac{2}{3} = \frac{2 \times 1}{3 \times 1}$$

Numbers named by fractions like $\frac{6}{12}$, $\frac{10}{15}$, and $\frac{15}{18}$ can be renamed in simplest form by dividing the numerator and denominator by their greatest common factor.

$$\frac{6}{12} = \frac{6 \div 6}{12 \div 6} = \frac{1}{2}$$

$$\frac{10}{15} = \frac{10 \div 5}{15 \div 5} = \frac{2}{3}$$

$$\frac{15}{18} = \frac{15 \div 3}{18 \div 3} = \frac{5}{6}$$

$$\frac{385}{66} = \frac{5 \times 7 \times 11}{2 \times 3 \times 11} = \frac{5 \times 7}{2 \times 3} = \frac{35}{6}$$

Names for rational numbers greater than 1 may be written in a variety of mixed forms.

$$3 + \frac{3}{4},\quad 2\frac{11}{8},\quad 1\frac{19}{8},\quad 3\frac{2}{5}$$

Mixed forms such as $3\frac{2}{5}$, $6\frac{3}{4}$, and $7\frac{5}{8}$ are each a simplest mixed form because

(1) the numerator of the fraction is smaller than the denominator, and

(2) the fraction is in simplest form.

Exercise Set 1

1. Replace n in each sentence by a numeral to make a true statement.

$4\frac{7}{8} = \frac{n}{8}$ $7\frac{5}{16} = \frac{n}{16}$ $46\frac{1}{4} = \frac{n}{4}$

$9\frac{1}{2} = \frac{n}{2}$ $13\frac{2}{3} = \frac{n}{3}$ $20\frac{7}{8} = \frac{n}{8}$

2. Write the following in the simplest mixed form.

$\frac{9}{6} =$ $5\frac{26}{6} =$ $\frac{18}{3} =$

$\frac{34}{8} =$ $7\frac{6}{3} =$ $3\frac{12}{8} =$

3. Replace n in each of the following to make the sentence true.

$5\frac{2}{3} = 4\frac{n}{3}$ $8\frac{5}{5} = 7\frac{n}{6}$ $57\frac{4}{9} = 56\frac{n}{9}$

$10\frac{3}{4} = 9\frac{n}{4}$ $14\frac{7}{8} = 13\frac{n}{8}$ $21\frac{9}{10} = 20\frac{n}{10}$

4. Write three more members for each set below.

Set A $= \left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \_\_, \_\_, \_\_\right\}$

Set B $= \left\{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \_\_, \_\_, \_\_\right\}$

Set C $= \left\{\frac{3}{4}, \frac{6}{8}, \frac{12}{16}, \_\_, \_\_, \_\_\right\}$

Set D $= \left\{\frac{2}{1}, \frac{4}{2}, \frac{10}{5}, \_\_, \_\_, \_\_\right\}$

5. Find the simplest fraction name for each rational number.

a. $\frac{9}{15}$
b. $\frac{6}{18}$
c. $\frac{3}{27}$
d. $\frac{21}{24}$
e. $\frac{12}{14}$
f. $\frac{8}{32}$
g. $\frac{56}{63}$
h. $\frac{28}{49}$
i. $\frac{27}{36}$
j. $\frac{24}{40}$
k. $\frac{12}{16}$
l. $\frac{8}{10}$

6. Copy and supply the missing numerator or denominator.

a. $\frac{1}{2} = \frac{\ }{16}$
b. $\frac{3}{4} = \frac{\ }{12}$
c. $\frac{7}{8} = \frac{21}{\ }$
d. $\frac{5}{6} = \frac{\ }{30}$
e. $\frac{2}{3} = \frac{10}{\ }$
f. $1\frac{4}{5} = 1\frac{\ }{10}$
g. $\frac{5}{8} = \frac{25}{\ }$
h. $\frac{2}{3} = \frac{\ }{9}$

COMMON DENOMINATOR

In your work with rational numbers, it is often convenient to work with fractions whose denominators are the same. If two or more fractions have the same denominator, we say they have a common denominator. Common denominators may be found for any two fractions by finding common multiples of both denominators.

Often your knowledge of the multiplication facts is all you need to do this. For greater, less familiar denominators you can find the least multiple common to both denominators.

Consider fractions with denominators 15 and 21. Suppose Set F is the set of multiples of 15, and Set T is the set of multiples of 21. Can you list every member of the set? How do you indicate that there are more members than it is possible to list?

Set F = {15, 30, 45, 60, 75, 90, 105, ...}
Set T = {21, 42, 63, 84, 105, ...}
$F \cap T$ = {105, ...}

Find the next seven members of Set F; of Set T. Find the next member of $F \cap T$.

Since 105 is the smallest number in Set $F \cap T$, it is called the least common multiple of 15 and 21. Any multiple of 105 is also a common multiple of 15 and 21. The product of 15 and 21, or of any two numbers, is always a common multiple of the numbers, but not always the least common multiple.

Complete factorization, or expressing a number as a product of primes, is the simplest method of finding a least common multiple for two numbers whose multiples you do not know.

Suppose the numbers are 12 and 21. Do you recall this convenient diagram you used in finding the least common multiple?

$$\overbrace{2 \times 2 \times \rlap{$\underbrace{}_{21}$}3}^{12} \times 7 = 84$$

84 is the least common multiple of 21 and 12. 84 is also the least common denominator of fractions with denominators 12 and 21.

Exercise Set 2

1. Rename the numbers named by each pair of fractions below by fractions which have the least common denominator. You should use just multiplication facts for these.

   a. $\frac{1}{6}, \frac{7}{9}$

   b. $\frac{5}{6}, \frac{2}{7}$

   c. $\frac{3}{4}, \frac{7}{8}$

   d. $\frac{3}{4}, \frac{5}{9}$

   e. $\frac{2}{9}, \frac{4}{7}$

   f. $\frac{6}{7}, \frac{3}{4}$

2. Which is the greater rational number?

   a. $\frac{3}{10}$ or $\frac{1}{3}$

   b. $\frac{5}{9}$ or $\frac{4}{7}$

   c. $\frac{2}{3}$ or $\frac{4}{7}$

   d. $\frac{3}{4}$ or $\frac{2}{5}$

3. Arrange in order from least to greatest.

   $\frac{2}{3}, \frac{5}{6}, \frac{1}{2}, \frac{4}{9}$

4. Write the first five members of the set of multiples for each number. Underline the least common multiple for each pair. Then write the first two members of the set of common multiples for each pair.

   a. 16, 12

   b. 20, 25

   c. 13, 26

   d. 15, 5

5. Name each pair of numbers by fractions with the least common denominator.

a. $\frac{5}{16}, \frac{7}{12}$

b. $\frac{31}{20}, \frac{9}{25}$

c. $\frac{8}{13}, \frac{7}{26}$

d. $\frac{18}{15}, \frac{9}{5}$

6. Find the least common multiple of each pair of numbers by using complete factorization.

a. 10, 14

b. 12, 20

c. 21, 30

d. 5, 16

e. 12, 14

f. 13, 15

7. Which is the greater rational number?

a. $\frac{5}{14}$ or $\frac{7}{12}$

b. $\frac{8}{15}$ or $\frac{7}{16}$

c. $\frac{11}{12}$ or $\frac{31}{32}$

d. $\frac{9}{10}$ or $\frac{13}{14}$

DECIMAL NAMES FOR RATIONAL NUMBERS

If you heard the words "three-tenths", you could write the fraction name $\frac{3}{10}$ or the decimal name 0.3.

Rational numbers whose fraction names have denominators 10, 100, or 1000 may be written directly as decimals.

$\frac{7}{10} = 0.7$ $\frac{39}{10} = 3.9$ $\frac{4325}{1000} = 4.325$

$\frac{2}{100} = 0.02$ $\frac{25}{1000} = 0.025$ $\frac{510}{100} = 5.10$

Some numbers like $\frac{1}{2}$ and $\frac{3}{4}$ must be renamed before they can be named by decimals.

$\frac{1}{2} = \frac{5}{10} = 0.5$ $\frac{3}{4} = \frac{75}{100} = 0.75$ $\frac{5}{8} = \frac{625}{1000} = 0.625$

Any decimal name for a rational number may be written directly as a fraction or mixed form.

$1.4 = 1\frac{4}{10}$ $2.007 = 2\frac{7}{1000}$

$0.59 = \frac{59}{100}$ $0.9 = \frac{9}{10}$

Recall that you can find a fraction name for a number, if you know a decimal name for it, in this way:

The digits in the decimal indicate the numerator.

The place value of the last digit on the right indicates the denominator.

For example, $3.457 = \frac{3457}{1000}$ $.0025 = \frac{25}{10,000}$

The fraction obtained in this way is called the fraction form of the decimal.

What is the fraction form of 1.4? of 2.007? of 0.59?

Exercise Set 3

1. Name in words the rational number named by each decimal below.

a. 1.2 b. 0.07 c. 0.435

2. Complete the following chart.

Fraction Name	Numerator	Denominator	Decimal Name
a. $\frac{5}{100}$	5	100	0.05
b. $\frac{76}{100}$			
c. $\frac{831}{1000}$			
d. $\frac{2408}{100}$			
e. $\frac{3012}{1000}$			
f. $\frac{134}{10}$			
g. $\frac{9}{1000}$			
h. $\frac{234}{100}$			

3. Match each fraction name in Column A with the decimal in Column B naming the same rational number.

Column A	Column B
a. $\frac{30}{100}$	g. 3.0
b. $\frac{3}{100}$	h. 0.30
c. $\frac{3}{1000}$	i. 0.003
d. $\frac{30}{10}$	j. 0.03
e. $\frac{30}{1000}$	k. 0.030
f. $\frac{300}{10}$	l. 30.0

4. Complete the chart below.

Decimal Name of Rational Number	Numerator of Fraction Form	Denominator of Fraction Form	Fraction Form
a. 0.42	42	100	$\frac{42}{100}$
b. 0.056			
c. 7.3			
d. 2.08			
e. 11.01			
f. 0.9			
g. 3.405			
h. 4.071			
i. 2.06			
j. 8.9			

5. Tell whether each of the rational numbers named below

> 1, < 1, or $= 1$.

a. 0.01
b. 0.1
c. 0.9

d. 1.1
e. 0.001
f. 1.0

g. 0.999
h. 1.010
i. 0.901

6. In each of the following, write a decimal name for a rational number, such that

a. $m > 0.11$ and $m < 0.2$ (Is more than one answer correct?)

b. $p > 0.009$ and $p < 0.009$

c. $t > 0.1$ and $t < 1.0$

d. $s > 1.1$ and $s < 1.9$

7. Arrange in order from least to greatest.

0.25, $2\frac{1}{4}$, 1, 0.02, 1.02, 2.002, 2.2

8. Write in expanded notation:

a. 4.9

b. 927.872

c. 56.63

9. Write the decimal numeral for:

a. $\frac{45}{1} + \frac{7}{10} + \frac{5}{100} + \frac{3}{1000}$

b. $(7 \times 10^2) + (2 \times 10^1) + (4 \times 1) + (9 \times \frac{1}{10^1}) + (7 \times \frac{1}{10^2})$

Exercise Set 4

1. Which of these fractions may be written directly as decimals? Write the decimals if you can write them directly.

a. $\frac{5}{20}$ c. $\frac{30}{60}$ e. $\frac{3}{4}$

b. $\frac{465}{100}$ d. $\frac{65}{10}$ f. $\frac{9}{1000}$

2. Rename each number so that it may be written directly as a decimal.

	Fraction Name	Multiply by	New Fraction Name	Decimal Name
a.	$\frac{12}{25}$	$\frac{4}{4}$	$\frac{48}{100}$	0.48
b.	$\frac{130}{500}$			
c.	$\frac{3}{2}$			
d.	$\frac{4}{5}$			
e.	$\frac{17}{20}$			
f.	$\frac{125}{250}$			
g.	$\frac{60}{50}$			
h.	$\frac{212}{200}$			
i.	$\frac{2}{125}$			

3. Which of the following numbers do not have a fraction name with denominator 10, 100, or 1000? Rename all others as decimals.

a. $\frac{6}{8}$ c. $\frac{100}{400}$ e. $\frac{3}{33}$

b. $\frac{40}{55}$ d. $\frac{150}{900}$ f. $\frac{8}{40}$

4. Complete:

Rational Number	Fraction Name	Decimal Name
one-fourth	$\frac{1}{4}$	0.25
one-half		
three-fourths		
four-twentieths		
three-fifths		
nine-tenths		

5. Match each fraction name in Column A with the correct decimal name in Column B.

Column A		Column B	
a.	$\frac{20}{25}$	h.	0.008
b.	$\frac{2}{250}$	i.	0.80
c.	$\frac{22}{25}$	j.	0.88
d.	$\frac{2}{25}$	k.	8.08
e.	$\frac{22}{250}$	l.	0.808
f.	$\frac{202}{250}$	m.	0.088
g.	$\frac{202}{25}$	n.	0.08

Exercise Set 5

1. a. Add 29.9 and 37.06

   b. Multiply $(\frac{2}{3})^2$ and $(\frac{3}{4})^2$

   c. Subtract 11.58 from 40

   d. Add $35\frac{5}{7}$ and $17\frac{3}{5}$

   e. Multiply 0.2 and 0.3

   f. Subtract $9\frac{7}{8}$ from $14\frac{1}{6}$

Read the following carefully. Express the relationships in each problem in a mathematical sentence. Solve, and write your answer in a complete sentence. Rational numbers may be represented by fractions or decimals.

2. One basketball player had an average of twenty-seven and nine-tenths points per game. A second player had an average of twenty-five and one-half points per game. Which player had the better average? By how many points was it better?

3. In a class of 32 children, there are 14 boys and 18 girls. 6 of the 14 boys are Scouts. 10 of the 18 girls are Campfire Girls. What part of the boys are Scouts? What part of the girls are Campfire Girls? What part of the class are Scouts? What part of the class are Campfire Girls? Which group represents the greater part of the class, Scouts or Campfire Girls?

4. In one part of the Amazon River Valley the average rainfall per month is twenty-one and four-tenths inches. What is the total amount of rainfall for the year?

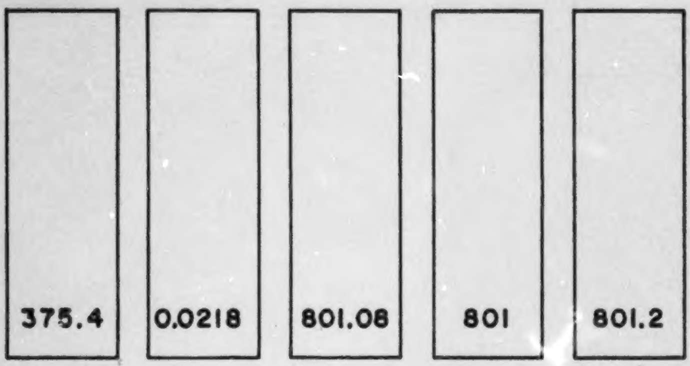
5. If a plane averages 560 miles an hour, how far will it travel in five and one-half hours?

6. On our vacation we averaged twenty-eight and seven-tenths miles per hour for an average of seven and one-half hours each day. What was the distance we traveled each day?

7. The average house fuse will allow 15 amperes of electricity to pass through it. If you connect a two and five-tenths amp heater and a ten amp refrigerator to it, how many more amps can the fuse carry without burning out?

8. A metric ton weighs two thousand two hundred four and six-tenths pounds. How many more pounds than the English ton is the metric ton?

9. In the school library, books are arranged on the shelves just as numbers are arranged on a number line. Call numbers on books become greater as you move to the right. How should the following books be arranged on the shelf?



10. The head is about $\frac{1}{5}$ of the height of a young boy. Jim is 60 inches tall. Will his head be above water when he stands in a pool marked 4 feet deep? Why?

PROPERTIES OF RATIONAL NUMBERS

Exploration

Add any two whole numbers.

Try three more pairs.

Did you always have a whole number to use as a sum?

Can you think of any two whole numbers whose sum is not a whole number?

Do you agree with this statement:

"The sum of any two whole numbers is always a whole number"?

1. Add the following pairs of rational numbers.

$$\begin{array}{llllllll} \text{a.} & 1\frac{2}{3} & \text{c.} & 4\frac{5}{6} & \text{e.} & 22\frac{3}{4} & \text{g.} & 58\frac{1}{2} \\ & \underline{\frac{1}{2}} & & \underline{2\frac{3}{8}} & & \underline{1\frac{1}{3}} & & \underline{2\frac{7}{10}} \\ \text{b.} & \frac{3}{4} & \text{d.} & 17\frac{1}{6} & \text{f.} & 5\frac{5}{6} & \text{h.} & 32\frac{6}{7} \\ & \underline{\frac{1}{3}} & & \underline{4\frac{5}{8}} & & \underline{\frac{3}{4}} & & \underline{17\frac{1}{2}} \end{array}$$

Was each sum a rational number?

Can you think of any two rational numbers whose sum is not a rational number?

Do you agree that the sum of any two rational numbers is a rational number?

2. Perform the following operations.

a. 432×56 c. 490×78 e. $3 \times \frac{5}{6}$ g. $2 \times 3\frac{2}{5}$

b. 708×9 d. 5600×43 f. $\frac{2}{3} \times \frac{4}{5}$ h. $\frac{3}{4} \times 4\frac{1}{2}$

When you multiplied two whole numbers, was the product always a whole number?

When you multiplied two rational numbers was the product always a rational number?

What generalization can you make about the product of two whole numbers? of two rational numbers?

3. Find the products of the following pairs of numbers.

a. $\frac{3}{4}$ and $\frac{4}{3}$

b. $1\frac{1}{2}$ and $\frac{2}{3}$

c. 5 and $\frac{1}{5}$

d. $\frac{1}{10}$ and 10

e. $1\frac{5}{6}$ and $\frac{6}{11}$

f. $\frac{7}{8}$ and $\frac{8}{7}$

What do you notice about the products you found?

4. Can you think of any two whole numbers whose product is 1? Did you think of 1 × 1? This is one more property of the interesting number 1:

When 1 is multiplied by 1, the product is 1. No other whole number has this property.

Every rational number, however, with the exception of zero, does possess this property:

For any rational number, except zero, there is another rational number such that the product of the numbers is 1. Such numbers are called reciprocals of each other.

5. Look back at exercise 3.

What do you notice about the numbers in each example?

Name the reciprocal of each of the following:

a. $\frac{4}{1}$ c. $\frac{3}{3}$ e. $\frac{6}{15}$

b. $\frac{7}{9}$ d. $\frac{2}{10}$ f. 2

6. Which product expressions below are names for 1?

a. $\frac{3}{4} \times \frac{4}{3}$ c. $5 \times \frac{1}{5}$ e. $1\frac{5}{6} \times \frac{4}{6}$

b. $1\frac{1}{2} \times \frac{2}{3}$ d. $\frac{1}{10} \times 1$ f. $\frac{2}{8} \times \frac{8}{7}$

7. Which of the following statements are true?

Explain your answer.

a. $7 + 4 = 4 + 7$

b. $17.53 + 34.7 = 34.7 + 17.53$

c. $3\frac{5}{8} + 12\frac{3}{4} = 12\frac{3}{4} + 3\frac{5}{8}$

d. $1.5 - 0.3 = 0.3 - 1.5$

e. $(3.7 \times 2.4) \times 6.8 = 3.7 \times (2.4 \times 6.8)$

f. $(\frac{1}{2} \times \frac{3}{4}) \times 1\frac{2}{3} = \frac{1}{2} \times (\frac{3}{4} \times 1\frac{2}{3})$

g. $16 \div (8 \div 2) = (16 \div 8) \div 2$

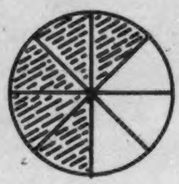
h. $2 \times (5 \times 3) = (2 \times 5) \times 3$

i. $7\frac{2}{5} - 3\frac{5}{6} = 3\frac{5}{6} - 7\frac{2}{5}$

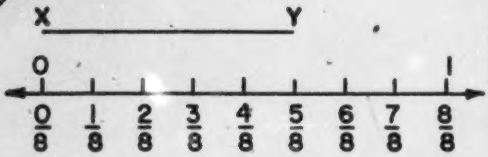
PICTURING RATIONAL NUMBERS

Rational numbers are used to describe the measure of a segment, region, or set in relation to a unit segment, unit region, or set.

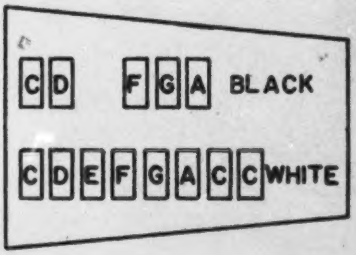
The following diagrams picture a region, a segment, and a set, each with measure $\frac{5}{8}$



A



B



C

$\frac{5}{8}$ of the circular region in figure A is shaded.

The measure of $\overline{XY}$ in figure B is $\frac{5}{8}$.

$\frac{5}{8}$ of the set of white keys on the song bells have matching black keys.

Just as 5 is another name for

$$1 + 1 + 1 + 1 + 1$$

or

$$5 \times 1,$$

$\frac{5}{8}$ is another name for

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

or

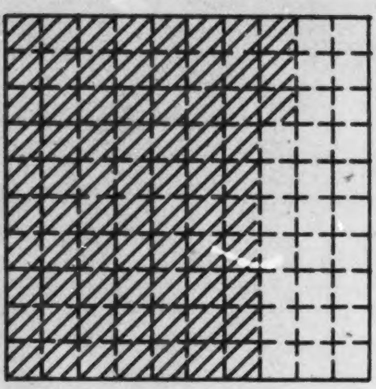
$$5 \times \frac{1}{8}$$

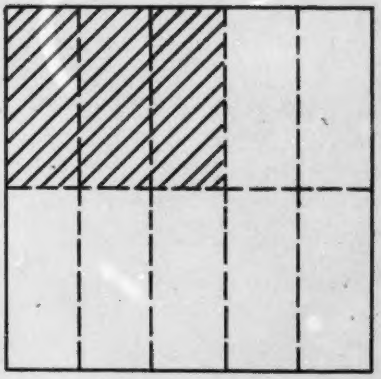
or

$$\frac{5}{8} \times 1.$$

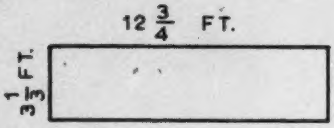
Exercise Set 6

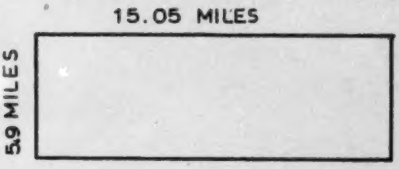
1. A and B are unit squares. Write both the fraction and decimal names that best describe the measure of each shaded region.



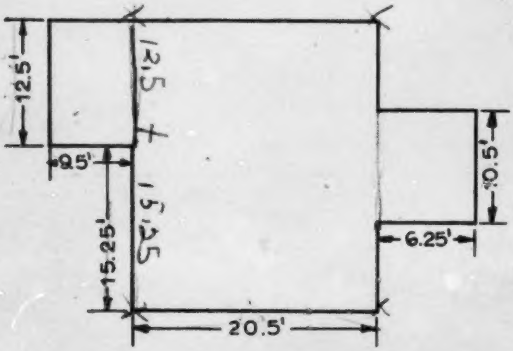


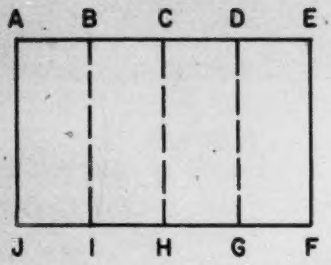
2. Find the perimeter and area of each rectangular region below.


12 $\frac{3}{4}$ FT.
3 $\frac{1}{3}$ FT.


15.05 MILES
5.9 MILES

3. Find the area of the floor of the house whose floor plan is drawn below.


12.5'
12.5
+
9.5'
15.25'
15.25
10.5'
6.25'
20.5'


A
B
C
D
E
J
I
H
G
F

1. AEFJ is a region separated into 4 congruent regions.
   If m AEFJ = 1,

   a. m ABIJ = ___ b. m ACHJ = ___ c. m ADGJ = ___

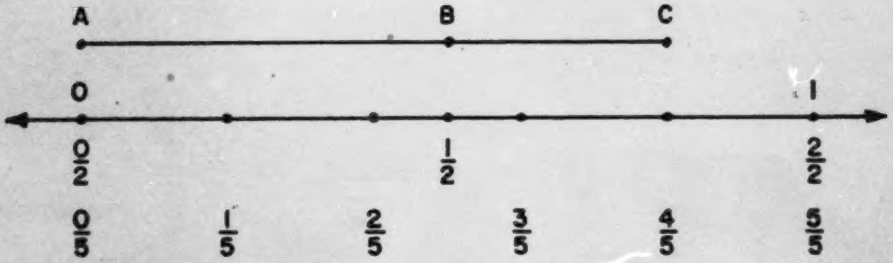
2. If m ADGJ = 1,

   a. m AEFJ = ___ b. m ACHJ = ___ c. m ABIJ = ___

3. If m ABIJ = 1,

   a. M AEFJ = ___ b. m ADGJ = ___ c. m ACHJ = ___

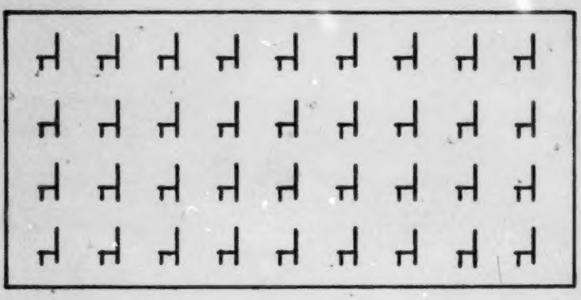
4. Write one addition and two subtraction sentences pictured by the diagram below. Write sentences using (a) fractions, and (b) decimals.


A
B
C
0
1
$\frac{0}{2}$
$\frac{1}{2}$
$\frac{2}{2}$
$\frac{0}{5}$
$\frac{1}{5}$
$\frac{2}{5}$
$\frac{3}{5}$
$\frac{4}{5}$
$\frac{5}{5}$

5. Illustrate the following problem with a number line drawing.

Tony made a chart to show the height in inches of all the children in his class. Allen told him he was $4\frac{2}{3}$ feet tall. How many inches did Tony record for Allen's height?

6. Below is a seating plan for a class of 36 children.



In how many other ways can you arrange the desks to make a series of equal rows? Draw a diagram similar to the one above to illustrate one arrangement and write the fraction name that best describes the measure of each row.

Exercise Set 7

1. Complete the following mathematical sentences.

   a. $(4.9 \times 3.52) + (4.9 - 3.52) = \_\_\_\_$

   b. $(7 - 4\frac{5}{6}) + (5\frac{1}{3} \times 2\frac{1}{2}) = \_\_\_\_$

   c. $(\frac{8}{15} \times 1\frac{1}{2}) - \frac{1}{5} + \_\_\_\_ = 7\frac{1}{10}$

   d. $(62.5 - 38.9) \times 4.3 = 25 + \_\_\_\_$

   e. $(1\frac{3}{4})^2 + (4 \times \frac{3}{7}) + \_\_\_\_ = 10$

   f. $(0.3 \times 0.3) + (20.6 - 12.93) - \_\_\_\_ = 5$

2. At one point, the river is 2.6 feet deep. At another, it is 5.1 feet deep. What is the difference in these two depths?

3. It takes $3\frac{1}{3}$ seconds for a word sent by telegraph from one station to reach a second station, 4 seconds more to reach the third, and 3.9 seconds more to reach the fourth. How long does it take to transmit the word from the first to the fourth station?

4. Ricky needed two 8.7 inch long axles and one 6 inch steering wheel pole for a toy racer he was making. He bought an iron rod 25 inches long. After he cut off the parts he needed, how many inches were left?

5. Plane A traveled 650 miles per hour for $3\frac{1}{2}$ hours. Plane B traveled 600 miles per hour for $3\frac{3}{4}$ hours. Which plane traveled the greater distance? How much greater was the distance?

6. What is the perimeter and what is the area of a rectangular room whose adjacent sides measure $24\frac{1}{2}$ feet by $16\frac{1}{2}$ feet?

7. Discoverer II was in orbit around the earth for a period of 90.5 minutes. Discoverer I was in orbit 95.569 minutes. What is the difference in time of the two orbital flights?

Exercise Set 8

Read the following problems carefully.

Express the relationships in each problem as a mathematical sentence.

1. Jane measured the tallest sunflower plant in her garden. It was 117 inches tall. What was its measurement in feet?

2. Bill's height is $\frac{3}{4}$ of Tom's. Tom is 6 feet tall. How tall is Bill?

3. Philip can throw a ball $\frac{4}{5}$ of the distance that Tim can. Tim can throw a ball 100 feet. How far can Philip throw it?

4. A piece of garden hose 17 feet long is divided into 2 equal lengths. What is the measure of each piece?

5. Mary and Florence bicycled to a picnic. They left home at the same time and traveled the same distance. Mary reached the picnic in $\frac{3}{4}$ of an hour. Florence reached it in $\frac{1}{2}$ of an hour. Which girl took more time? How much more?

6. Wildwood and Hillcrest Schools are having a debate. Each school has two speakers on its team. The debate will last one hour. The time will be divided equally among the speakers. What part of an hour will each speaker have?

7. The area of a rectangular flower bed is 24 square feet. The measure of one side is 8 feet. What is the measure of the adjacent side?

8. In January, 1961, 4.8 inches of rain fell in New Orleans. In February, 3.1 inches fell. How much rain did New Orleans have in the two months?

9. It takes Jupiter 11.862 earth years to orbit the sun. It takes Saturn 29.458 earth years to orbit the sun. What is the difference in earth years between the times required for Jupiter and Saturn to orbit the sun?

10. In 1959 a man-eating white shark measuring $16\frac{5}{6}$ feet was caught by rod and reel in Australia, setting a new record. A year later, a blue shark measuring $11\frac{1}{2}$ feet was caught by rod and reel in the United States, setting a world record also. How much greater was the length of the white shark?

GETTING READY FOR DIVISION OF RATIONAL NUMBERS

Exploration

You have studied multiplication of rational numbers. Now you will study division of rational numbers.

You know that division with whole numbers is a process of finding an unknown factor of a product when one factor is known. We shall think of division with rational numbers as having the same meaning. That is, the sentence

$$\frac{5}{8} \div \frac{2}{3} = \frac{15}{16}$$

means $$\frac{2}{3} \times \frac{15}{16} = \frac{5}{8}$$

Is this multiplication sentence true?

Until now, you have used only factors which were counting numbers. In thinking about division of rational numbers we shall also consider rational numbers as factors of a product which is a rational number. For example, in the sentences

$$\frac{5}{8} \times n = 7 \qquad \text{or} \qquad 7 \div \frac{5}{8} = n$$

we shall speak of 7 as the product, $\frac{5}{8}$ as one factor, and n as the other factor. We also can use the language of division and call 7 the dividend, $\frac{5}{8}$ the divisor and n the quotient.

In the next set of exercises use what you have learned about products of rational numbers and about the operation of division to get ready to divide by a rational number.

Exercise Set 9

1. Find a fraction name for each product expression.

a. $\frac{2}{3} \times \frac{3}{2} = n$

b. $\frac{5}{9} \times \frac{9}{5} = n$

c. $8 \times \frac{1}{8} = n$

d. $\frac{2}{15} \times 7\frac{1}{2} = n$

e. $1\frac{1}{3} \times \frac{3}{4} = n$

f. $\frac{10}{19} \times 1\frac{9}{10} = n$

g. $\frac{237}{459} \times \frac{459}{237} = n$

h. $\frac{57}{59} \times 1\frac{2}{57} = n$

i. $3\frac{1}{7} \times \frac{7}{22} = n$

2. In example 1, each product is the number _____. In each product expression, one factor is the _____ of the other.

3. For each sentence, find a number n which makes the sentence true.

a. $\frac{4}{3} \times n = 1$

b. $n \times \frac{5}{8} = 1$

c. $\frac{4}{15} \times n = \frac{4}{15}$

d. $\frac{1}{10} \times n = 1$

e. $n \times \frac{1}{13} = 1$

f. $n \times \frac{23}{8} = \frac{23}{8}$

g. $(\frac{10}{13} \times \frac{13}{10}) \times \frac{7}{8} = n$

h. $(\frac{8}{9} \times \frac{9}{8}) \times n = \frac{7}{15}$

i. $(\frac{2}{3} \times n) \times \frac{3}{5} = \frac{3}{5}$

j. $(9 \times n) \times \frac{15}{7} = \frac{15}{7}$

k. $(n \times \frac{1}{7}) \times \frac{5}{11} = \frac{5}{11}$

l. $\frac{8}{9} \times (n \times \frac{3}{4}) = \frac{8}{9}$

m. $\frac{7}{5} \times (\frac{9}{10} \times \frac{10}{9}) = n$

n. $n \times (\frac{7}{11} \times \frac{11}{7}) = \frac{5}{13}$

4. Here are some division sentences. Write the relationship in each sentence as a multiplication sentence.

a. $20 \div n = 4$

b. $65 \div 13 = n$

c. $42 \div 14 = n$

d. $42 \div n = 14$

e. $n \div 14 = 42$

f. $19 \div n = 19$

g. $1 \div n = 19$

h. $1 \div 19 = n$

i. $n \div \frac{2}{3} = \frac{3}{2}$

5. For each sentence in exercise 4, find a number n which makes the sentence true. (Use your multiplication sentences if you prefer to do so.)

6. Rewrite each division sentence as a multiplication sentence which states the same relationship.

a. $15 \div n = (\frac{2}{3} \times \frac{3}{2})$

b. $15 \div (\frac{9}{7} \times \frac{7}{9}) = n$

c. $\frac{4}{9} \div (\frac{5}{8} \times \frac{8}{5}) = n$

d. $n \div (\frac{7}{3} \times \frac{3}{7}) = \frac{2}{9}$

e. $7\frac{1}{2} \div n = (\frac{11}{4} \times \frac{4}{11})$

f. $1 \div \frac{13}{3} = n$

g. $(\frac{10}{7} \times \frac{7}{10}) \div n = 2\frac{3}{4}$

h. $6\frac{2}{3} \div (\frac{6}{7} \times \frac{7}{6}) = n$

i. $n \div (\frac{7}{12} \times 1\frac{5}{7}) = 5\frac{1}{2}$

j. $n \div 8\frac{1}{2} = 2$

k. $n \div \frac{7}{9} = \frac{2}{3}$

l. $\frac{8}{15} \div \frac{2}{3} = n$

m. $(\frac{8}{15} \times \frac{3}{2}) \div n = (\frac{2}{3} \times \frac{3}{2})$

7. For each sentence you wrote in exercise 6, find the number n which makes your multiplication sentence true.

8. For each multiplication sentence below, write two division sentences which state the same relationship.

a. $\frac{2}{3} \times \frac{3}{2} = 1$

b. $\frac{5}{9} \times \frac{9}{5} = 1$

c. $8 \times \frac{1}{8} = 1$

d. $\frac{2}{15} \times 7\frac{1}{2} = 1$

e. $1\frac{1}{3} \times \frac{3}{4} = 1$

f. $\frac{10}{19} \times 1\frac{9}{10} = 1$

g. $\frac{237}{459} \times \frac{459}{237} = 1$

h. $\frac{57}{59} \times 1\frac{2}{57} = 1$

i. $3\frac{1}{7} \times \frac{7}{22} = 1$

9. When a product is the number 1, and one factor is a rational number, the unknown factor is the ______ of the first factor.

We write:

$$1 \div \frac{a}{b} = \frac{b}{a}, \text{ when } a \neq 0 \text{ and } b \neq 0.$$

Exploration

You know that $6 \times 5 = 30$. The " = " means that "6×5" and "30" are two names for the same number.

If $6 \times 5 = 30$, what number is $2 \times 6 \times 5$? Is it 2×30?

The Associative Property tells us that

$$2 \times (6 \times 5) = (2 \times 6) \times 5.$$

So $2 \times 6 \times 5$ can be thought of as $2 \times (6 \times 5)$ or as $(2 \times 6) \times 5$. If we choose to think of it as $2 \times (6 \times 5)$, then

$$2 \times 6 \times 5 = 2 \times 30.$$

Now consider this sentence:

$$7 \times n = 42.$$

If $7 \times n$ and 42 are two names for the same number, you know that $n = 6$. No other number for n will make $7 \times n = 42$ a true sentence.

If $7 \times n = 42$, what about $5 \times (7 \times n)$? How must you operate on 42 to get a product which names the same number as $5 \times (7 \times n)$?

If $7 \times n = 42$

then $5 \times 7 \times n = \_\_\_ \times 42$

Complete these sentences. Remember the meaning of " = ".

1. If $3 \times 8 = 24$

then $2 \times 3 \times 8 = \_\_\_ \times 24$

2. If $48 = 6 \times 8$

then $\_\_\_ \times 48 = 3 \times 6 \times 8$

3. If $21 = 7 \times 3$

then $21 \times \frac{3}{5} = 7 \times 3 \times \_\_\_$

4. If $3 \times 15 = 45$

then $3 \times 15 \times \frac{5}{8} = 45 \times \_\_\_$

5. If $6 \times n = 18$

then $2 \times 6 \times n = \_\_\_ \times 18$

6. If $\frac{2}{3} \times r = 12$

then $\frac{3}{2} \times \frac{2}{3} \times r = \_\_\_ \times 12$

7. If $n \times \frac{3}{4} = 15$

then $n \times \frac{3}{4} \times \frac{6}{7} = \_\_\_ \times \frac{6}{7}$

8. If $\frac{5}{7} \times n = 40$

then $\frac{5}{7} \times n \times \frac{7}{5} = 40 \times \_\_\_$

9. If $36 = \frac{9}{10} \times k$

then $\frac{10}{9} \times 36 = \_\_\_ \times \frac{9}{10} \times k$

10. If $\frac{2}{9} = \frac{4}{5} \times n$

then $\frac{9}{2} \times \frac{2}{9} = \frac{9}{2} \times \_\_\_ \times \_\_\_$

11. $\_\_\_ \times \frac{3}{5} \times n = 1 \times n,$ if n is *any* rational number.

12. $\frac{2}{3} \times \_\_\_ \times n = 1 \times n$

13. $n \times \frac{19}{5} \times \_\_\_ = n \times 1$

14. $1 \times n = \frac{4}{5} \times \_\_\_ \times n$

15. $n \times 1 = n \times \frac{10}{9} \times \_\_\_$

16. If $\frac{7}{8} \times n = 35$

then $\frac{8}{7} \times \frac{7}{8} \times n = \_\_\_ \times 35$

and $\_\_\_ \times n = \_\_\_ \times 35$

and $n = \_\_\_$

17. If $n \times \frac{3}{4} = \frac{15}{7}$

then $n \times \frac{3}{4} \times \frac{4}{3} = \frac{15}{7} \times \underline{\qquad}$

and $n \times 1 = \frac{15}{7} \times \underline{\qquad}$

and $n = \underline{\qquad}$

18. If $n \times \frac{8}{5} = \frac{16}{7}$

then $n \times \frac{8}{5} \times \frac{5}{8} = \underline{\qquad} \times \frac{5}{8}$

and $n \times \underline{\qquad} = \underline{\qquad}$

and $n = \underline{\qquad}$

19. If $\frac{9}{2} = \frac{2}{7} \times n$

then $\frac{7}{2} \times \frac{9}{2} = \frac{7}{2} \times \underline{\qquad} \times n$

and $\frac{7}{2} \times \frac{9}{2} = \underline{\qquad} \times n$

and $\underline{\qquad} = n$

20. If $\frac{8}{3} \times n = \frac{7}{12}$

then $\frac{3}{8} \times \frac{8}{3} \times n = \underline{\qquad} \times \frac{7}{12}$

and $\underline{\qquad} \times n = \underline{\qquad} \times \frac{7}{12}$

and $n = \underline{\qquad}$

DIVISION BY A RATIONAL NUMBER

Exploration

You know how to find an unknown factor of the number 1 when you know the other factor. Now consider finding an unknown factor of any rational number.

Think about this sentence:

$$\frac{7}{2} \div \frac{3}{4} = n$$

1. What multiplication sentence states the same relationship?

2. Complete this sentence:

$$\underline{\quad\quad} \times \frac{3}{4} \times n = 1 \times n$$

Your answers to exercises 1 and 2 suggest a way to find $1 \times n$, or n.

Your multiplication sentence was

$$\frac{3}{4} \times n = \frac{7}{2}$$

You wish to find n, or $1 \times n$.

Your answer to exercise 2 shows that

$$\frac{4}{3} \times \frac{3}{4} \times n = 1 \times n.$$

This suggests multiplying $\frac{3}{4} \times n$ by $\frac{4}{3}$.

3. Look at these sentences. Can you complete them?

a. $\frac{3}{4} \times n = \frac{7}{2}$

b. $\frac{4}{3} \times \frac{3}{4} \times n = \underline{\qquad} \times \frac{7}{2}$

c. $\underline{\qquad} \times n = \underline{\qquad} \times \frac{7}{2}$

d. $n = \underline{\qquad} \times \frac{7}{2}$, or $\underline{\qquad}$,

The simplest fraction name is ____.

e. So $\frac{7}{2} \div \frac{3}{4} = \underline{\qquad}$

Now think about this example:

4. $\frac{5}{4} \div \frac{2}{7} = n$

Use the multiplication sentence

$$n \times \frac{2}{7} = \frac{5}{4}$$

a. You have a number for $n \times \frac{2}{7}$. You want to find $n \times 1$.

$$n \times 1 = n \times \frac{2}{7} \times \underline{\qquad}$$

b. Now use the result in exercise a.

Multiply $n \times \frac{2}{7}$ by ____

c. $n \times \frac{2}{7} \times \underline{\qquad} = \frac{5}{4} \times \underline{\qquad}$

d. $n \times \underline{\qquad} = \frac{5}{4} \times \underline{\qquad}$

e. $n = \underline{\qquad}$

f. $\frac{5}{4} \div \frac{2}{7} = \underline{\qquad}$

Now look at this example.

$$3\frac{1}{3} \div 1\frac{3}{8} = n$$

Rename the numbers: $\frac{10}{3} \div \frac{11}{8} = n$

State the multiplication sentence:

$$n \times \frac{11}{8} = \frac{10}{3}$$

To get $n \times 1$, multiply $(n \times \frac{11}{8})$ by $\frac{8}{11}$:

$$(n \times \frac{11}{8}) \times \frac{8}{11} = \frac{10}{3} \times \frac{8}{11}$$

Use the Associative Property:

$$n \times (\frac{11}{8} \times \frac{8}{11}) = \frac{10}{3} \times \frac{8}{11}$$

$\frac{11}{8} \times \frac{8}{11} = 1$ $\qquad n \times 1 = \frac{10}{3} \times \frac{8}{11}$

Use the Property of One: $n = \frac{10}{3} \times \frac{8}{11}$

Now you have a product expression:

$$n = \frac{10 \times 8}{3 \times 11} = \frac{80}{33} = \frac{66}{33} + \frac{14}{33}$$

$$n = 2\frac{14}{33}$$

So $3\frac{1}{3} \div 1\frac{3}{8} = 2\frac{14}{33}$

Check: Does $1\frac{3}{8} \times 2\frac{14}{33} = 3\frac{1}{3}$?

$$\frac{11}{8} \times \frac{80}{33} = \frac{11 \times 80}{8 \times 33} = \frac{11 \times 8 \times 10}{8 \times 11 \times 3} = 3\frac{1}{3}$$

Now try this example: Write out your work as shown above.

$$\frac{16}{5} \div \frac{8}{3} = n$$

Exercise Set 10

Copy the work for these division examples and complete each sentence to make it true.

1. $\frac{7}{8} \div \frac{2}{3} = n$

$n \times \frac{2}{3} = \frac{7}{8}$

$(n \times \frac{2}{3}) \times \underline{\qquad} = \frac{7}{8} \times \underline{\qquad}$ (same number in both blanks)

$n \times (\frac{2}{3} \times \underline{\qquad}) = \frac{7}{8} \times \underline{\qquad}$ (same number as before)

$n \times 1 = \frac{7}{8} \times \underline{\qquad}$ (same number as before)

$n = \frac{7}{8} \times \underline{\qquad}$

$n = \underline{\qquad}$ (simplify product expression)

$\frac{7}{8} \div \frac{2}{3} = \underline{\qquad}$

2. $\frac{15}{4} \div n = \frac{5}{2}$

$n \times \frac{5}{2} = \frac{15}{4}$

$(n \times \frac{5}{2}) \times \underline{\qquad} = \frac{15}{4} \times \underline{\qquad}$ (Same number in both blanks)

$n \times (\frac{5}{2} \times \underline{\qquad}) = \frac{15}{4} \times \underline{\qquad}$

$n \times \underline{\qquad} = \frac{15}{4} \times \underline{\qquad}$ (Different numbers)

$n = \frac{15}{4} \times \underline{\qquad}$

$n = \underline{\qquad}$

$\frac{15}{4} \div \underline{\qquad} = \frac{5}{2}$

3. Are the numbers suggested for n correct? Check by multiplying.

a. $\frac{6}{7} \div \frac{8}{5} = n$

$n = \frac{15}{28}$

b. $3\frac{5}{7} \div n = 1\frac{3}{5}$

$n = 2\frac{1}{4}$

Find a number n which makes each sentence true.

Write your work as shown in exercises 1 and 2.

4. $9\frac{3}{4} \div \frac{13}{7} = n$

5. $2\frac{5}{6} \div n = 5\frac{1}{2}$

6. $1\frac{3}{4} \div 3\frac{1}{2} = n$

7. $2\frac{3}{10} \div n = \frac{4}{5}$

COMPUTING QUOTIENTS OF RATIONAL NUMBERS

Exploration

You have learned how to divide by a rational number, using the meaning of division and the properties of multiplication of rational numbers. Now see whether there is a short way to compute quotients.

1. Explain lines b to f of this example.

a. $\frac{7}{8} \div \frac{3}{4} = n$

b. $n \times \frac{3}{4} = \frac{7}{8}$

c. $(n \times \frac{3}{4}) \times \frac{4}{3} = \frac{7}{8} \times \frac{4}{3}$

d. $n \times (\frac{3}{4} \times \frac{4}{3}) = \frac{7}{8} \times \frac{4}{3}$

e. $n \times 1 = \frac{7}{8} \times \frac{4}{3}$

f. $n = \frac{7}{8} \times \frac{4}{3}$

2. Now compare lines a and f. From these lines you can see that

$$\frac{7}{8} \div \frac{3}{4} = n = \frac{7}{8} \times \frac{4}{3}, \quad \text{or}$$

$$\frac{7}{8} \div \frac{3}{4} = \frac{7}{8} \times \frac{4}{3}$$

What do you observe?

3. Now explain each line of this example.

a. $4\frac{2}{3} \div 10\frac{1}{2} = n$

b. $\frac{14}{3} \div \frac{21}{2} = n$

c. $n \times \frac{21}{2} = \frac{14}{3}$

d. $(n \times \frac{21}{2}) \times \frac{2}{21} = \frac{14}{3} \times \frac{2}{21}$

e. $n \times (\frac{21}{2} \times \frac{2}{21}) = \frac{14}{3} \times \frac{2}{21}$

f. $n \times 1 = \frac{14}{3} \times \frac{2}{21}$

g. $n = \frac{14}{3} \times \frac{2}{21}$

h. $\frac{14}{3} \div \frac{21}{2} = \frac{14}{3} \times \frac{2}{21}$ (From b and g)

4. What relation does line h suggest?

5. Suppose you apply your observations in exercises 2 and 4 and write this sentence.

$$\frac{5}{8} \div \frac{3}{4} = \frac{5}{8} \times \frac{4}{3}.$$

You can test the correctness of this sentence by using the meaning of division. That is,

Does $(\frac{5}{8} \times \frac{3}{4}) \times \frac{3}{4} = \frac{5}{8}$?

Does $\frac{20}{24} \times \frac{3}{4} = \frac{5}{8}$?

Does $\frac{60}{96} = \frac{5}{8}$?

$$\frac{60}{96} = \frac{2 \times 2 \times 3 \times 5}{2 \times 2 \times 2 \times 2 \times 2 \times 3} = \frac{5}{2 \times 2 \times 2} = \frac{5}{8}$$

So the quotient $\frac{5}{8} \div \frac{3}{4}$ is the same number as the product $\frac{5}{8} \times \frac{4}{3}$.

Do you see an easier way to show that $(\frac{5}{8} \times \frac{4}{3}) \times \frac{3}{4} = \frac{5}{8}$?

6. Exercises 2, 4, and 5 suggest that

a. $\frac{3}{4} \div \frac{9}{10} = \frac{3}{4} \times$ ____.

b. $\frac{12}{5} \div \frac{7}{16} =$ ____ × ____.

c. $\frac{1}{7} \div \frac{1}{10} =$ ____ × ____.

d. $3\frac{5}{8} \div 2\frac{1}{4} = \frac{29}{8} \times$ ____.

e. $9 \div 6\frac{1}{5} =$ ____ × ____.

f. $10\frac{3}{4} \div 12 =$ ____ × ____.

7. Write the multiplication sentence which states the same relationship as each division sentence.

a. $\frac{17}{5} \div \frac{6}{7} = \frac{17}{5} \times \frac{7}{6}$

b. $8\frac{3}{4} \div 2\frac{1}{2} = \frac{35}{4} \times \frac{2}{5}$

c. $\frac{4}{21} \div \frac{3}{7} = \frac{21}{4} \times \frac{3}{7}$

d. $5\frac{3}{8} \div 9\frac{1}{3} = \frac{43}{8} \times \frac{3}{28}$

8. Test each multiplication sentence you wrote for exercise 7 to see whether or not the sentence is true.

Was any sentence false? If so, can you change any numbers to make a true sentence?

9. Write a statement describing a short way to divide by a rational number.

10. If $b \neq 0$, $c \neq 0$, $d \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{?}{?}$

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}}$$

Exercise Set 11

Use $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ to find simple names for these quotients.

1. $\frac{5}{8} \div \frac{3}{7}$	6. $7\frac{1}{2} \div 12$
2. $3\frac{5}{8} \div 2\frac{1}{3}$	7. $5280 \div 16\frac{1}{2}$
3. $7 \div \frac{3}{5}$	8. $320 \div 144$
4. $\frac{24}{16} \div \frac{6}{4}$	9. $16\frac{1}{2} \div 3$
5. $\frac{9}{4} \div \frac{4}{9}$	10. $14 \div 2\frac{1}{2}$

11. Match each quotient expression in Column A with the product expression which names the same number in Column B.

Column A	Column B
a. $\frac{3}{4} \div \frac{4}{3}$	e. $\frac{4}{3} \times \frac{4}{3}$
b. $\frac{3}{4} \div \frac{3}{4}$	f. $\frac{3}{4} \times \frac{3}{4}$
c. $\frac{4}{3} \div \frac{4}{3}$	g. $\frac{3}{4} \times \frac{4}{3}$
d. $\frac{4}{3} \div \frac{3}{4}$	h. $\frac{4}{3} \times \frac{3}{4}$

12. Complete:

a. $6 \div \frac{3}{4} = 6 \times \_\_\_ = \frac{24}{3} = 8$

b. $\frac{5}{3} \div \frac{5}{6} = \frac{5}{3} \times \_\_\_ = \frac{30}{15} = \_\_\_$

c. $\frac{3}{4} \div \_\_\_ = \frac{3}{4} \times 8 = \frac{24}{4} = \_\_\_$

d. $\frac{1}{2} \div \frac{1}{2} = \frac{1}{2} \times \_\_\_ = \_\_\_$

e. $\frac{12}{8} \div \frac{3}{2} = \frac{12}{8} \times \_\_\_ = \_\_\_$

Find a fraction or mixed form for n which makes each sentence true.

13. $9\frac{7}{8} \div 2\frac{3}{4} = n$

14. $12\frac{1}{2} \div 50 = n$

15. $10 \div 3\frac{1}{7} = n$

16. $1 \div 2\frac{2}{10} = n$

17. $8\frac{1}{2} \div \frac{3}{5} = n$

18. $19\frac{1}{3} \div 3 = n$

19. $3\frac{4}{15} \div 4\frac{2}{3} = n$

20. $100 \div 2\frac{1}{4} = n$

21. $2\frac{3}{10} \div 4 = n$

22. $16\frac{4}{5} \div 4\frac{9}{10} = n$

23. Write out work for exercise 13 as shown in exercise 1 of Exercise Set 10.

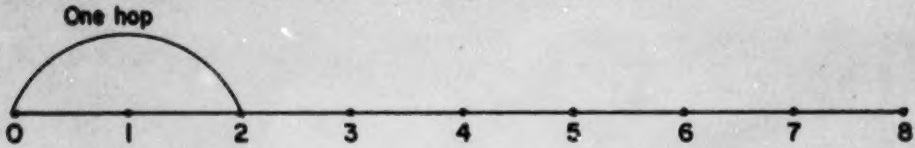
24. Write out exercise 19 as shown in exercise 1 of Exercise Set 10.

PROBLEMS SOLVED BY DIVISION

Exploration

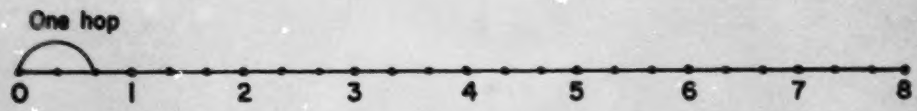
You have learned how to divide by a rational number. Now think about what this means when you have a problem to solve in which the numbers are rational numbers.

1. A mechanical toy moves by hops. In each hop it covers 2 feet. How many hops will it take to cover 8 feet?



You know it will take 4 hops. How would you show this on the diagram? What mathematical operation on the number 8 and 2 gives you the answer 4? The mathematical sentence is $8 \div 2 = n$ or $2 \times n = 8$.

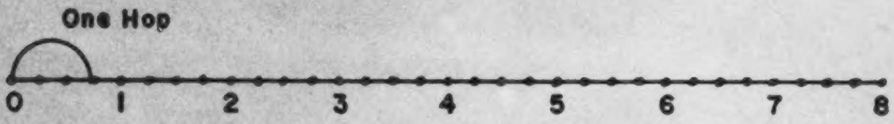
2. Now suppose you have a small copy of the toy. This one covers $\frac{2}{3}$ foot in each hop. How many hops does it make to cover 8 feet?



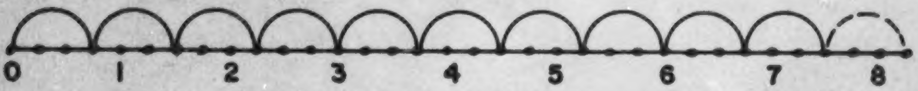
Count on the diagram to find the answer. (Notice each unit segment is separated into three congruent segments.) Did you get 12?

Is this the same kind of problem as exercise 1? What is the mathematical sentence for it?

3. Another of these toys hops $\frac{3}{4}$ foot. How many hops will it take to cover 8 feet?



Count on the diagram to find the answer. (Notice each unit segment is separated into four congruent segments.) Is the answer a whole number?



Ten hops take the toy $7\frac{1}{2}$ feet. The next hop will take it beyond 8 feet. What "part of a hop" will take it just to 8 feet?

What is the mathematical sentence for this problem?

Now look at the mathematical sentences and the solutions you have found from the diagrams.

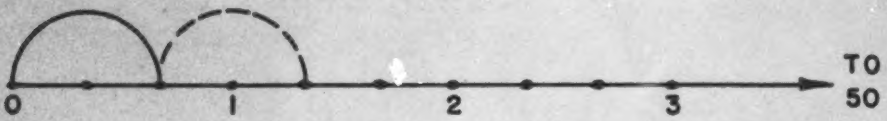
1. $8 \div 2 = n$ $n = 4$
2. $8 \div \frac{2}{3} = n$ $n = 12$
3. $8 \div \frac{3}{4} = n$ $n = 10\frac{2}{3}$

4. Find names for these quotients by using $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

a. $8 \div 2$ b. $8 \div \frac{2}{3}$ c. $8 \div \frac{3}{4}$

Do your results agree with your answers for exercises 1-3?

5. Look again at exercise 2. If this toy had to go 50 feet, it would be awkward to find the answer from a diagram. Look at the part of the diagram which represents the first foot.



The toy covers $\frac{2}{3}$ foot in its first hop. The second hop will take it past 1. What "part of a hop" will take it to 1? To cover 1 foot, the toy takes $1\frac{1}{2}$ hops, or $\frac{3}{2}$ hops. If you know this, how can you find how many hops it will take for 2 feet? for 5 feet? for 50 feet? Check the following on the diagram:

1 foot	$\frac{3}{2}$ hops
2 feet	$2 \times \frac{3}{2}$
3 feet	$3 \times \frac{3}{2}$

6. Write the mathematical sentence for this problem. If a toy hops $\frac{2}{3}$ feet in one hop, how many hops must it make to go 50 feet?

7. Solve the sentence $50 \div \frac{2}{3} = n$. Use $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

8. $50 \div \frac{2}{3} = \frac{50}{1} \times \frac{3}{2}$. In the diagram in exercise 5, what does $\frac{3}{2}$ represent?

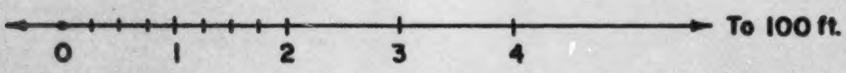
9. Look again at exercise 3. The sentence for this problem was

$$8 \div \frac{3}{4} = n$$

$$8 \div \frac{3}{4} = \frac{8}{1} \times \frac{4}{3}.$$

What does the number $\frac{4}{3}$ represent in the diagram?

10. a. Trace the number line below and use it to make the first part of a diagram to represent the hops of a toy which hops $1\frac{3}{4}$ feet per hop and has 100 feet to go.



b. Write the mathematical sentence showing the relationship of the 100 feet, the $1\frac{3}{4}$ feet, and the number of hops.

c. Solve the mathematical sentence using $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

What does $\frac{d}{c}$ represent in the diagram?

Exercise Set 12

For each problem, make part or all of a line diagram. Then write the mathematical sentence. Solve the sentence and answer the question.

1. A store had 75 pounds of candy. The owner decided to put it in bags containing $\frac{5}{8}$ pounds each. How many bags could he fill?

2. Tom watched a bug crawl up a wall 9 feet high. It climbed $\frac{5}{6}$ foot in a minute. At that rate, how long would it take the bug to reach the top?

3. Joan's mother had $1\frac{3}{4}$ dozen eggs. Joan had a dessert recipe which called for $\frac{5}{12}$ dozen. She wanted to make as much dessert as possible for a party. How many recipes could she make if she used all the eggs?

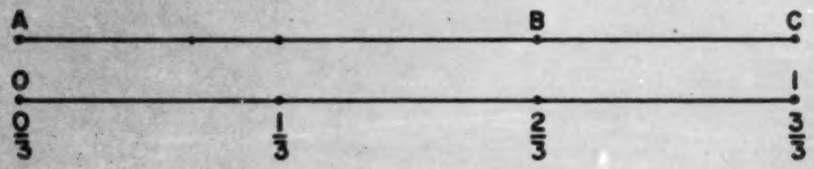
4. A frozen food company has 50 pounds of peas. How many $\frac{5}{8}$ pound packages can they make?

5. You have a piece of cardboard 30 inches long and 10 inches wide. How many 10-inch strips can you cut from the piece if you make the strips $1\frac{3}{4}$ inches wide?

DIVISION ON THE NUMBER LINE

Exploration

You can picture division on the number line if you think of two different scales on the same line. Consider $1 \div \frac{2}{3}$.

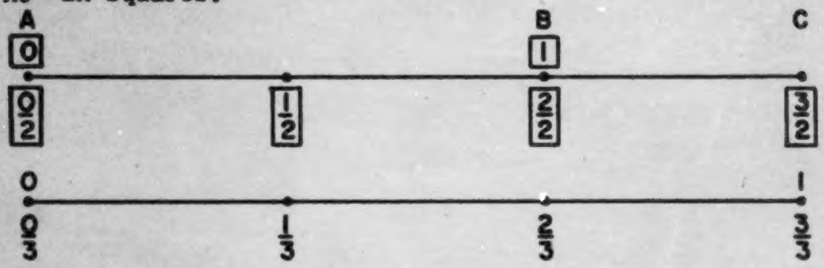


In the diagram, what segment has measure 1?

What segment has measure $\frac{2}{3}$?

Now think about a second scale on the number line in which $\overline{AB}$ has measure 1.

On the diagram below labels for the new scale are written on $\overline{AC}$ in squares.



How many times can the new unit segment $(\overline{AB})$ be laid off on $\overline{AC}$?

The picture shows that $m\overline{AC} \div m\overline{AB} = \frac{3}{2}$

In the original scale $1 \div \frac{2}{3} = \frac{3}{2}$

In the new scale, $\frac{3}{2} \div 1 = \frac{3}{2}$

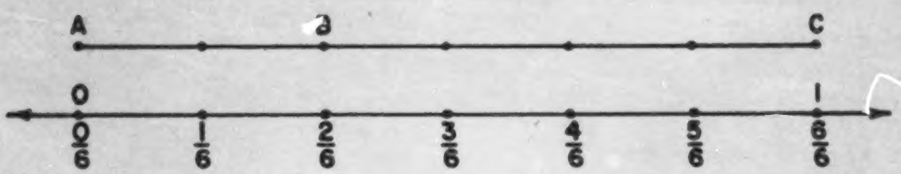
The picture also shows that $m\overline{AB} \div m\overline{AC} = \frac{2}{3}$

In the original scale, $\frac{2}{3} \div 1 = \frac{2}{3}$

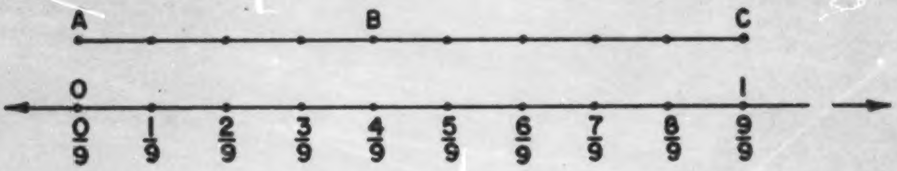
In the new scale, $1 \div \frac{3}{2} = \frac{2}{3}$

What division sentence is pictured on each number line below? In each, consider $m\overline{AC}$ as a product and $m\overline{AB}$ as the known factor.

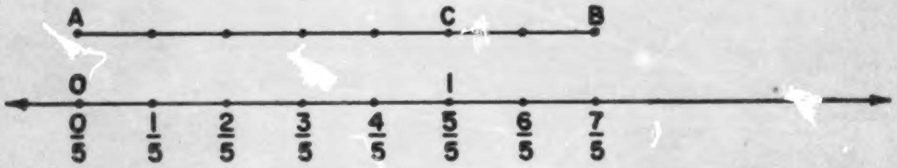
1.


A
B
C
0
1
0/6
1/6
2/6
3/6
4/6
5/6
6/6

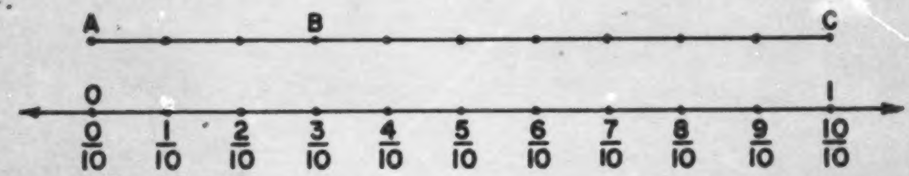
2.


A
B
C
0
1
0/9
1/9
2/9
3/9
4/9
5/9
6/9
7/9
8/9
9/9

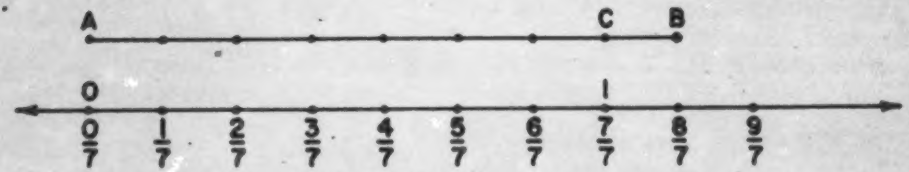
3.


A
C
B
0
1
0/5
1/5
2/5
3/5
4/5
5/5
6/5
7/5

4.


A
B
C
0
1
0/10
1/10
2/10
3/10
4/10
5/10
6/10
7/10
8/10
9/10
10/10

5.


A
C
B
0
1
0/7
1/7
2/7
3/7
4/7
5/7
6/7
7/7
8/7
9/7

60

Write a division sentence for each problem. Picture each one on a number line.

5. You have a measuring cup full of orange juice. You serve it in small glasses, each holding $\frac{3}{8}$ cup. How many servings are there in 1 cup?

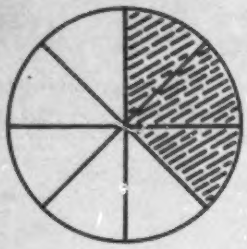
6. You have 1 yard of ribbon which you cut to make badges, $\frac{2}{9}$ of a yard for each badge. You have enough for how many badges?

7. You have one cup of milk to make candy. Your recipe calls for $\frac{3}{4}$ cup. How many recipes can you make if you use all of the milk?

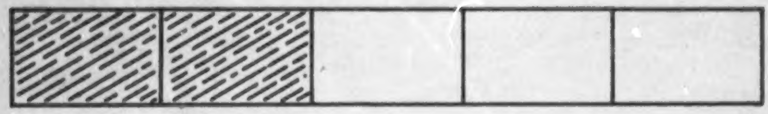
8. You have 1 mile to walk. You can walk $\frac{5}{2}$ mile in an hour. How long will it take you to walk 1 mile?

9. In many countries distances are measured in kilometers, rather than in miles. One kilometer is about $\frac{5}{8}$ mile. If you walk 1 mile, about how many kilometers have you walked?

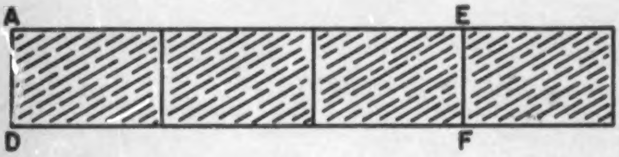
10. How many shaded regions are needed to cover the unit circular region?



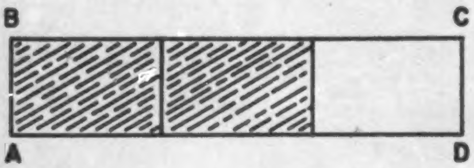
11. How many shaded regions will it take to cover the unit rectangular region?



12. How many shaded regions are needed to cover region AEFD?



13. If the measure of region ABCD is 1, how many shaded regions will cover a region whose measure is 6?



Exercise Set 13

1. You have a 90 mile trip to make. How many miles must you travel per hour, on the average, to finish in

   a. 3 hours? What operation did you use?

   b. 2 hours?

   c. $2\frac{1}{2}$ hours? What operation?

   d. $1\frac{3}{4}$ hours?

   e. Suppose you are a racing driver. If you drive a 90-mile course in $\frac{9}{10}$ of an hour, what is your average number of miles per hour?

2. Express each measurement in the unit named.

   a. $34\frac{1}{2}$ ounces is _____ pounds.

   b. $8\frac{1}{2}$ inches is _____ feet.

   c. $9\frac{7}{10}$ feet is _______ yards.

   d. $9\frac{1}{4}$ pints is ______ quarts.

   e. $24\frac{3}{5}$ seconds is _____ minutes.

3. You know that the measure of a rectangular region is the product of the measures of two adjacent sides.

Find in the table below the missing measure for each region.

Rectangular Region	Measure of First Side	Measure of Second Side	Measure of Region
A	$2\frac{5}{8}$		$3\frac{7}{16}$
B	$15\frac{3}{4}$	$12\frac{1}{2}$	
C		$\frac{7}{12}$	$\frac{3}{8}$
D	$\frac{3}{4}$		$1\frac{4}{5}$
E		$3\frac{7}{8}$	$6\frac{1}{2}$

Write a mathematical sentence for each problem. Be sure to answer the question in a complete sentence.

4. A peanut vendor at the ball park bagged 500 pounds of peanuts, putting about $\frac{3}{8}$ pound in each bag. How many bags could he fill from 1 pound? From the 500 pounds?

5. One of the satellites takes $1\frac{3}{5}$ hours to travel around the earth. How many trips does it make in a day?

6. The Sault Sainte Marie Canal is 1.2 miles long and the Welland Canal is 27.6 miles long.

The Welland Canal is how many times as long as the Sault Sainte Marie Canal? (Use fraction names to compute.)

7. Plans are made to use jacks to raise the temple in Egypt called Abu Simbel 200 feet, in stages of .04 inches. At this rate, how many stages will be required?

8. A club bought 50 yards of material to make towels for a bazaar. If each towel requires $\frac{7}{8}$ yard of material, how towels can be made?

9. A factory workman can complete one article in $2\frac{1}{2}$ minutes. How many articles can he complete in 8 hours if there is no loss of time?

10. If a girl can knit $1\frac{3}{4}$ inches of a scarf in one hour, how many hours will it take to complete a 35 inch scarf?

Exercise Set 14

1. Name each number by a decimal.

a. $\frac{17}{100}$ c. $\frac{6}{20}$ e. $\frac{49}{50}$ g. $1\frac{11}{10}$

b. $\frac{1}{4}$ d. $\frac{1}{2}$ f. $\frac{75}{250}$ h. $3\frac{6}{5}$

2. Find the simplest fraction name for each number.

a. 0.12 c. 1.010 e. 3.19 g. 9.9

b. 0.07 d. 0.101 f. 0.006 h. 4.02

3. Write in the simplest fraction or mixed form.

a. $\frac{90}{10}$ c. $\frac{430}{100}$ e. $\frac{8}{24}$ g. $\frac{6}{18}$

b. $2\frac{7}{6}$ d. $\frac{12}{5}$ f. $\frac{609}{100}$ h. $\frac{2000}{1000}$

4. Name each of these numbers by a fraction. Use the simplest fraction name.

a. $5\frac{1}{2}$ c. 0.20 e. $1\frac{5}{4}$ g. $7\frac{3}{8}$

b. $6\frac{6}{6}$ d. 0.750 f. 4.1 h. 12.01

5. Arrange in order from greatest to least.

1.1; $\frac{1}{10}$; 2.11; 1.01; $\frac{3}{1000}$; 0.101; 0.11

6. Write the reciprocal of each number.

a. $\frac{1}{4}$ c. $\frac{7}{10}$ e. $\frac{1}{1}$ g. $\frac{13}{9}$

b. 2 d. $1\frac{5}{9}$ f. $3\frac{2}{7}$ h. $15\frac{4}{7}$

7. Add each pair of numbers and express the result in simplest form.

a. $\begin{array}{r} \frac{2}{3} \\ \frac{1}{2} \\ \hline \end{array}$ c. $\begin{array}{r} \frac{3}{4} \\ \frac{2}{3} \\ \hline \end{array}$ e. $\begin{array}{r} 1\frac{5}{6} \\ 2\frac{3}{8} \\ \hline \end{array}$ g. $\begin{array}{r} 7\frac{1}{6} \\ 4\frac{5}{8} \\ \hline \end{array}$

b. $\begin{array}{r} 12\frac{3}{4} \\ 1\frac{1}{3} \\ \hline \end{array}$ d. $\begin{array}{r} 4\frac{5}{6} \\ \frac{1}{4} \\ \hline \end{array}$ f. $\begin{array}{r} 8\frac{1}{8} \\ 2\frac{7}{10} \\ \hline \end{array}$ h. $\begin{array}{r} 2\frac{6}{7} \\ 7\frac{1}{2} \\ \hline \end{array}$

8. Subtract each pair of numbers and express the result in the simplest fraction or mixed form.

a. $\begin{array}{r} \frac{12}{10} \\ 1\frac{1}{5} \\ \hline \end{array}$ c. $\begin{array}{r} 1\frac{1}{3} \\ 1\frac{1}{8} \\ \hline \end{array}$ e. $\begin{array}{r} 4\frac{3}{3} \\ 3\frac{5}{5} \\ \hline \end{array}$ g. $\begin{array}{r} 14\frac{1}{9} \\ 5\frac{1}{2} \\ \hline \end{array}$

b. $\begin{array}{r} 1\frac{7}{10} \\ 1\frac{5}{12} \\ \hline \end{array}$ d. $\begin{array}{r} 23\frac{3}{6} \\ 14\frac{3}{5} \\ \hline \end{array}$ f. $\begin{array}{r} 1\frac{1}{2} \\ \frac{3}{4} \\ \hline \end{array}$ h. $\begin{array}{r} 6\frac{2}{3} \\ 5\frac{8}{7} \\ \hline \end{array}$

9. Find a simplest fraction name or mixed form for each product expression.

a. $2\frac{1}{2} \times \frac{5}{8}$ d. $\frac{5}{6} \times 8\frac{1}{4}$ g. $\frac{1}{9} \times \frac{1}{9}$

b. $3\frac{1}{4} \times 4\frac{1}{3}$ e. $2\frac{2}{5} \times 2\frac{2}{7}$ h. $5\frac{3}{7} \times 1\frac{5}{9}$

c. $9\frac{7}{8} \times \frac{1}{2}$ f. $\frac{12}{2} \times \frac{10}{9}$ i. $\frac{19}{3} \times \frac{21}{1}$

10. Express each quotient in simplest form.

a. $7 \div \frac{3}{4}$ d. $9\frac{2}{3} \div 2\frac{3}{4}$ g. $5\frac{1}{2} \div 2$

b. $\frac{3}{4} \div \frac{2}{3}$ e. $14 \div 2\frac{2}{3}$ h. $21 \div \frac{2}{3}$

c. $\frac{7}{8} \div 4$ f. $\frac{11}{12} \div 1\frac{1}{2}$ i. $7\frac{5}{6} \div 3\frac{2}{3}$

j. Write out your work as shown in exercise 1 of Exercise Set 10:

$$\frac{11}{5} \div \frac{7}{8}$$

11. Complete with " >," " <," or " = ":

a. $\frac{7}{8} \times \frac{3}{4}$ ______ $\frac{1}{4} + \frac{7}{12}$

b. $5\frac{1}{2} - 4\frac{11}{12}$ ______ $2\frac{3}{4} \div 11$

c. $7\frac{1}{3} \div 2\frac{1}{5}$ ______ $5 - 1\frac{2}{3}$

d. $2\frac{1}{2} + 3\frac{3}{8}$ ______ $3\frac{3}{8} - 2\frac{5}{16}$

e. 8.4×0.74 ______ 0.75×8.4

f. $\frac{5}{6} \div \frac{5}{12}$ ______ $17.8 \div 8.9$

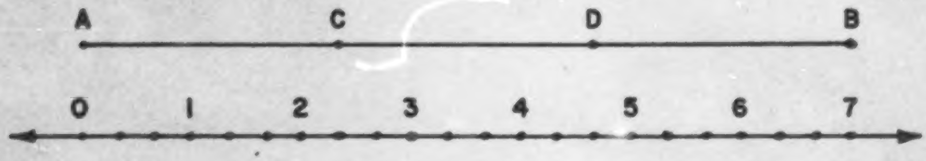
RATIONAL NUMBERS AS QUOTIENTS OF WHOLE NUMBERS

Exploration

The idea of a rational number and the operation of division of counting numbers are closely related.

Consider this question:

If a 7-inch segment is separated into 3 congruent segments, how long is each segment?



Since each segment is $\frac{1}{3}$ of $\overline{AB}$, its length in inches is

$$\frac{1}{3} \text{ of } 7, \text{ or } \frac{1}{3} \times \frac{7}{1}, \text{ or } \frac{7}{3}$$

The language of the problem suggests that we can think of it in another way, as $7 \div 3$. Since whole numbers are also rational numbers,

$$\begin{aligned} 7 \div 3 &= \frac{7}{1} \div \frac{3}{1} \\ &= \frac{7}{1} \times \frac{1}{3} = \frac{7}{3} \end{aligned}$$

Thus, one meaning for $\frac{a}{b}$, when a and b are whole numbers $(b \neq 0)$, is $a \div b$.

Test this with $\frac{8}{5}$.

$$\begin{aligned} \frac{8}{5} = \frac{8 \times 1}{1 \times 5} = \frac{8}{1} \times \frac{1}{5} &= \frac{8}{1} \div \frac{5}{1} \quad \left(\text{Since } \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}\right) \\ &= 8 \div 5. \end{aligned}$$

1. Express these quotients of counting numbers as rational numbers. Show your work as in exercise a.

   a. $18 \div 6 = \frac{18}{1} \times \frac{1}{6} = \frac{18}{6}$

   b. $28 \div 9$

   c. $5 \div 247$

   d. $46 \div 93$

   e. $1729 \div 1000$

   f. $a \div b$

2. Express these rational numbers as quotients of counting numbers. Show your work.

   a. $\frac{4}{7} = \frac{4}{1} \times \frac{1}{7} = \frac{4}{1} \div \frac{7}{1} = 4 \div 7.$

   b. $\frac{6}{13}$

   c. $\frac{13}{8}$

   d. $\frac{65}{127}$

   e. $\frac{79}{12}$

   f. $\frac{m}{n}$

3. Complete the following sentences.

   a. If a 2 hour period is divided into 5 periods of equal length, the length of each period is _____ hours.

   b. A recipe calls for 3 cups of milk. To make half the recipe, _____ cups of milk should be used.

   c. A man divided his garden into 5 parts of equal area. Each piece had area $\frac{576}{5}$ square feet. The area of the garden was _____ square feet.

   d. A rectangular field is 34 yards wide and has area 1700 square yards. The length of the field is _____ yards.

   e. The quotient of two counting numbers is always a _______________ number.

By extending the operation of division to rational numbers we have also learned a new way to express the division process for whole numbers. In the fourth and fifth grades, you learned that to divide 31 by 3 meant to find n and r in the sentence

$$31 = (3 \times n) + r$$

so that n and r are whole numbers and $r < 3$.

If it turned out that $r = 0$ as in

$$33 = (3 \times n) + r$$

then we were actually finding an unknown factor.

Now we know that there is always a rational number missing factor. In other words there is a rational number p so that

$$31 \div 3 = p.$$

How is p related to n and r?

$$31 = (3 \times \overset{n}{10}) + \overset{r}{1}$$

$$31 \div 3 = \frac{31}{3} = \frac{(3 \times 10) + 1}{3} = \frac{3 \times 10}{3} + \frac{1}{3} = 10 + \frac{1}{3}$$

So $$p = \underset{n}{10}\overset{r}{\frac{1}{3}}$$

Here is another example: $146 \div 27$

$$\begin{array}{r} 5 \\ 27\,\overline{)\,146} \\ \underline{135} \\ 11 \end{array} \qquad\qquad 146 = (27 \times \underset{\substack{\uparrow \\ n}}{5}) + \underset{\substack{\uparrow \\ r}}{11}$$

Now we can express the division process using rational numbers.

$$\frac{146}{27} = \frac{(27 \times 5) + 11}{27} = \frac{27 \times 5}{27} + \frac{11}{27} = 5 + \frac{11}{27}$$

$$146 \div 27 = \underset{\substack{\uparrow \\ n}}{5}\overset{\swarrow r}{\frac{11}{27}}$$

We also can express a simplest mixed form as a quotient of whole numbers.

$$6\frac{7}{18} = 6 + \frac{7}{18} = \frac{18 \times 6}{18} + \frac{7}{18} = \frac{(18 \times 6) + 7}{18} = \frac{115}{18}$$

So, $6\frac{7}{18} = 115 \div 18$

Another way to express this relation is the sentence

$$115 = (18 \times 6) + 7.$$

Exercise Set 15

1. Use the relation between division and rational numbers to show why each sentence is true.

Example: $(8 \times 39) \div 13 = 8 \times 3$

This is true because

$$(8 \times 39) \div 13 = \frac{8 \times 39}{13} = \frac{8 \times 3 \times 13}{13} = 8 \times 3$$

a. $(5 \times 12) \div 3 = 5 \times 4$

b. $(7 \times 15) \div 5 = 7 \times 3$

c. $(17 \times 2) \div 6 = 17 \div 3$

d. $(3 \times 4) \div 20 = 3 \div 5$

e. $(7^5 \div 7^2) = 7^3$

f. $(5 \times 6^3) \div 6^2 = 5 \times 6$

g. $(5 \times 6^2) \div 6^3 = 5 \div 6$

h. $(11 \times \frac{2}{3}) = (11 \div 3) \times 2$

i. $(11 \times \frac{2}{3}) = (11 \times 2) \div 3$

j. $(n \times .7) = (n \times 7) \div 10$

k. $(n \times .76) = (n \times 76) \div 100$

2. Express the quotient of each pair of numbers below in simplest mixed form. Also express the relation between the numbers in the form $a = b \times n + r$

Example: 44, 6

$$44 \div 6 = \frac{44}{6} = 7\frac{2}{6} = 7\frac{1}{3}$$

$$44 = (6 \times 7) + 2$$

a. 46, 7
b. 98, 13
c. 68, 12
d. 55, 8
e. 104, 13
f. 365, 7
g. 130, 16

3. The perimeter of a square is 17 inches. What is the length of one side?

4. Ann's mother divided a quart (32 ounces) of lemonade among 5 children. How many ounces did each child get?

5. In Nevada, Joan's family drove 420 miles in 8 hours. What was their rate in miles per hour?

6. BRAINTWISTER. Suppose m, n, and p are counting numbers.

a. Translate this sentence into the language of fractions.

$$(m + n) \div p = (m \div p) + (n \div p).$$

b. In grade 4, we found that the sentence was true if p was a common factor of m and n as in

$$96 \div 8 = 80 \div 8 + 16 \div 8.$$

What did we call this property?

c. If we use divison of rational numbers, is the sentence in (a) true for any counting numbers m, n, p?

Show why or why not.

EXTENDING THE SYSTEM OF FRACTIONS

Exploration

You have seen that $\frac{7}{3}$ is the quotient $7 \div 3$. Now look at the quotient $3 \div .4$

Even though $.4$ is not a whole number, it is convenient to write the division expression $3 \div .4$ as the fraction $\frac{3}{.4}$.

This is just like inventing a new word by telling what it is to mean. We simply agree that the new symbol $\frac{3}{.4}$ names the result of operating on 3 and $.4$ by division.

The sentence

$$\frac{3}{.4} = 3 \div .4$$

tells the meaning of $\frac{3}{.4}$.

We will use this meaning for all numerals of the form $\frac{a}{b}$ with rational numbers for a and b $(b \neq 0)$.

$$\frac{6.2}{2.7} = 6.2 \div 2.7$$

$$\frac{3.4}{.6} = 3.4 \div .6.$$

This meaning for fractions agrees with what we already know about fractions using whole numbers.

$$\frac{7}{4} = 7 \div 4$$

If our "new" fractions did not have the same properties as our "old" fractions, there would be no reason to use such symbols for quotients. In fact, it would be confusing to do so. We can show that these properties are still true, and we can use them in division problems. As examples we give two of these properties of all fractions and show why they are true.

For a fraction with whole number numerator and denominator you know that

$$\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{3 \times 3}{4 \times 3} = \ldots.$$

We might express this property as

Property (I)

$$\frac{a}{b} = \frac{a \times m}{b \times m},$$

when a, b, and m are whole numbers $(b \neq 0, m \neq 0)$

Test it, for rational numbers, in this example:

Is it true that

$$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{3} \times \frac{3}{4}} ?$$

$$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2}$$

$$\frac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{3} \times \frac{3}{4}} = \frac{\frac{3}{8}}{\frac{3}{12}} = \frac{3}{8} \div \frac{3}{12} = \frac{3}{8} \times \frac{12}{3} = \frac{3 \times 12}{8 \times 3} = \frac{12}{8} = \frac{3}{2}.$$

So

$$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{3} \quad \frac{3}{4}}$$

Look at this example in another way to see whether this property will always be true.

Does $\dfrac{\frac{1}{2}}{\frac{1}{3}} = \dfrac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{3} \times \frac{3}{4}}$

Suppose $\dfrac{\frac{1}{2}}{\frac{1}{3}} = n$

Then $n = \frac{1}{2} \div \frac{1}{3}$ (Fraction symbol indicates division)

$n \times \frac{1}{3} = \frac{1}{2}$ (We can rewrite a division sentence as a multiplication sentence)

$(n \times \frac{1}{3}) \times \frac{3}{4} = \frac{1}{2} \times \frac{3}{4}$ (Since $(n \times \frac{1}{3})$ and $\frac{1}{2}$ name the same number)

$n \times (\frac{1}{3} \times \frac{3}{4}) = \frac{1}{2} \times \frac{3}{4}$ (Associative Property)

$n = (\frac{1}{2} \times \frac{3}{4}) \div (\frac{1}{3} \times \frac{3}{4})$ (We can rewrite a multiplication sentence as a division sentence)

$n = \dfrac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{3} \times \frac{3}{4}}$ (A quotient can be rewritten as a fraction)

So $\dfrac{\frac{1}{2}}{\frac{1}{3}} = \dfrac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{3} \times \frac{3}{4}}$

So Property (I) was true in this case.

This same way of thinking can be used to show that Property I is always true if a, b, and m are rational numbers and b and m are not 0.

Use Property I to find other fraction names for these numbers. Multiply the numerator and denominator by the number suggested for m.

1. $\dfrac{\frac{3}{8}}{\frac{5}{4}}$ $m = 8$

2. $\dfrac{\frac{7}{3}}{\frac{5}{6}}$ $m = 6$

3. $\frac{1.5}{2.7}$ $m = 10$

4. $\frac{700}{800}$ $m = \frac{1}{100}$

5. To rename $\dfrac{\frac{1}{2}}{\frac{1}{3}}$ as a fraction with whole number numerator and denominator, what number should you use for m? Show that your choice for m is correct.

6. To rename $\dfrac{\frac{3}{8}}{\frac{5}{4}}$ so the denominator of the fraction name will be 1, what number should you use for m? Show that your choice for m is correct.

7. Rename the number in exercise 1 by dividing $\frac{3}{8}$ by $\frac{5}{4}$.

8. Rename the number in exercise 2 by dividing $\frac{7}{3}$ by $\frac{5}{6}$.

9. Do your answers for exercises 1 and 7 name the same number? What about your answers for exercises 2 and 8?

Exercise Set 16

Use Property I $\left(\frac{a}{b} = \frac{a \times m}{b \times m}\right)$ to rename each number by a fraction with whole number numerator and denominator.

1. $\dfrac{\frac{5}{8}}{\frac{3}{4}}$

2. $\dfrac{\frac{7}{5}}{\frac{8}{3}}$

3. $\dfrac{\frac{3}{2}}{\frac{7}{10}}$

4. $\dfrac{6}{3\frac{1}{7}}$

5. $\dfrac{.9}{.07}$

6. $\dfrac{1.25}{2.3}$

Use Property I $\left(\frac{a}{b} = \frac{a \times m}{b \times m}\right)$ to show that each of these sentences is true.

7. $\dfrac{\frac{3}{10}}{\frac{7}{10}} = \dfrac{3}{7}$

8. $\dfrac{.45}{.7} = \dfrac{45}{70}$

9. $\dfrac{2.47}{.6} = \dfrac{24.7}{6}$

10. $\dfrac{\frac{1}{2}}{\frac{1}{3}} = \dfrac{3}{2}$

11. $\dfrac{4}{5} = \dfrac{\frac{4}{5}}{1}$

12. $\dfrac{\frac{2}{5}}{.67} = \dfrac{40}{67}$

13. $\dfrac{.4}{2\frac{3}{5}} = \dfrac{2}{13}$

14. $\dfrac{2\frac{1}{7}}{2\frac{1}{7}} = 1$

Exploration

You have seen that the property $\frac{a}{b} = \frac{a \times m}{b \times m}$ is true when a, b, and m are rational numbers (not 0).

Here is another property of "whole number fractions"

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5}, \qquad \frac{7}{8} \times \frac{3}{4} = \frac{7 \times 3}{8 \times 4}.$$

We shall call this property "Property II"

Property (II) We write this property as:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \quad (\text{If } b \neq 0 \text{ and } d \neq 0)$$

Is this property still true if a, b, c, and d are rational numbers but not necessarily whole numbers?

As an illustration, is this sentence true?

$$\frac{\frac{1}{2}}{\frac{2}{3}} \times \frac{3}{\frac{1}{5}} = \frac{\frac{1}{2} \times 3}{\frac{2}{3} \times \frac{1}{5}} = \frac{\frac{3}{2}}{\frac{2}{15}}$$

To see that the sentence is correct, we remember:

$$\frac{\frac{1}{2}}{\frac{2}{3}} = \frac{1}{2} \div \frac{2}{3} = \frac{1}{2} \times \frac{3}{2}$$

$$\frac{3}{\frac{1}{5}} = 3 \div \frac{1}{5} = 3 \times 5$$

Then $$\frac{\frac{1}{2}}{\frac{2}{3}} \times \frac{3}{\frac{1}{5}} = \left(\frac{1}{2} \times \frac{3}{2}\right) \times (3 \times 5)$$

$$= \left(\frac{1}{2} \times 3\right) \times \left(\frac{3}{2} \times 5\right) \qquad \text{(Why?)}$$

$$= \frac{3}{2} \times \frac{15}{2}$$

$$= \frac{3}{2} \div \frac{2}{15} \qquad \text{(Why?)}$$

$$= \frac{\frac{3}{2}}{\frac{2}{15}}$$

This way of thinking can show that Property II

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

is true if a, b, c and d are rational numbers ($b \neq 0$, $d \neq 0$).

Use Properties (I) and (II) to show that each of these sentences is true:

1. $\frac{4}{5} \times \frac{\frac{1}{3}}{\frac{1}{6}} = \frac{8}{5}$

2. $\frac{.6}{.7} \times \frac{.3}{.2} = \frac{9}{7}$

3. $3 \times \frac{\frac{2}{5}}{\frac{1}{4}} = 4\frac{4}{5}$

Exercise Set 17

1. Since every fraction with rational numerator and denominator names a rational number, it must have a simplest fraction name and a simplest mixed form

   Find each of these for the fractions below.

   Example: $\frac{5.5}{1.7}$

   $\frac{5.5}{1.7} = \frac{5.5 \times 10}{1.7 \times 10} = \frac{55}{17}$ (simplest fraction name)

   $\frac{55}{17} = 3\frac{4}{17}$ (simplest mixed form)

   a. $\frac{6.3}{5}$

   b. $\frac{\frac{3}{4}}{\frac{2}{3}}$

   c. $\frac{1.8}{.5}$

   d. $\frac{2\frac{1}{2}}{.6}$

2. Because the middle "bar" in $\frac{\frac{3}{4}}{\frac{2}{3}}$ means division we can write $\frac{\frac{3}{4}}{\frac{2}{3}} = \frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{9}{8}$. Use this method to find simplest forms for the following:

   a. $\frac{\frac{1}{2}}{\frac{1}{3}}$

   b. $\frac{3}{\frac{1}{4}}$

   c. $\frac{1\frac{1}{2}}{\frac{2}{3}}$

   d. $\frac{\frac{5}{8}}{\frac{8}{5}}$

DIVISION OF RATIONAL NUMBERS NAMED BY DECIMALS

Exploration

Suppose that we are using only whole numbers and have a division problem like this:

Divide 1267 by 240.

What do we find? We have a division process for finding a relation between 1267 and 240. If 1267 is a multiple of 240, the process will give the decimal numeral for the whole number $1267 \div 240$. If 1267 is not a multiple of 240, the process will find what multiple of 240 less than 1267 is closest to 1267. We show our process in this way.

$$\begin{array}{r} 5 \\ 240\,\overline{)\,1267} \\ \underline{1200} \\ 67 \end{array}$$

The relation is expressed by this sentence:

$$1267 = (240 \times 5) + 67.$$

We can not write a division sentence because no whole number n makes these sentences true:

$$1267 = 240 \times n.$$

$$1267 \div 240 = n.$$

If we use rational numbers then we can write such sentences. They will be

$$1267 = 240 \times (5\tfrac{67}{240}) \quad \text{or}$$

$$1267 \div 240 = 5\tfrac{67}{240}.$$

This is the usual way in which the instructions

Divide 1267 by 240

are followed. We use the division process to find a name for the rational number $1267 \div 240$. The division process as you have learned it gives a mixed form for the quotient.

Use the division process to find a mixed form for each of these:

1. $187 \div 24$
2. $277 \div 31$
3. $207 \div 23$
4. $875 \div 43$

Can we use the division process to get mixed form names for quotients of any rational numbers whose decimals are given? Suppose our problem is

$$12.67 \div 2.4 = n$$

(n is to be named by a mixed form.)

The division sentence $12.67 \div 2.4 = n$ might be written as

$$\frac{12.67}{2.4} = n$$

Then we could write

$$\frac{12.67}{2.4} = \frac{12.67 \times 100}{2.4 \times 100} = \frac{1267}{240} = 1267 \div 240 = 5\frac{67}{240}$$

We know how to translate any fraction like $\frac{1267}{240}$ into mixed form.

The trick is to rename the number by a "whole number" fraction.

Example: $\frac{23.5}{1.74} = \frac{23.5 \times 10^2}{1.74 \times 10^2} = \frac{2350}{174}$

$$\begin{array}{r} 13 \\ 174\,\overline{)\,2350} \\ \underline{174} \\ 610 \\ \underline{522} \\ 88 \end{array} \qquad \frac{23.5}{17.4} = 13\frac{88}{174} = 13\frac{44}{87}$$

Express each of these quotients in simplest mixed form:

5. $\frac{15.6}{2.3}$

6. $\frac{258.7}{23}$

7. $438 \div 14.9$

Many times it is convenient to express the quotient of two rational numbers as a decimal. If the numbers are named as decimals, then it is natural to compute in decimal language. One difficulty is that as yet we do not have a decimal name for every rational number. As yet we have no decimals for $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, etc. A second difficulty is that our process may not give the decimal name even when there is one. Here is an example:

$$\frac{15.6}{2} = \frac{156}{20} \qquad \begin{array}{r} 7 \\ 20\,\overline{)\,156} \\ \underline{140} \\ 16 \end{array} \qquad \frac{15.6}{2} = 7\frac{16}{20} = 7\frac{80}{100} = 7.80 \text{ or } 7.8.$$

We had to find a decimal name after finding a mixed form. The division process alone did not give us the decimal 7.8.

In the next sections we will find out how to extend the division process to find decimal names.

Exercise Set 18

1. Which are other names for the first number in each row?

a. $\frac{25.6}{.14}$; $\frac{2560}{14}$, $\frac{1280}{7}$, $\frac{2.56}{1.4}$.

b. $\frac{13.54}{.5}$; 13.54×2, $13.54 \div 5$, $27\frac{2}{25}$

c. $16 \div 7\frac{2}{5}$; $\frac{16}{7\frac{2}{5}}$, $\frac{80}{37}$, $\frac{.08}{.037}$

d. $\frac{2.64}{.32}$; $\frac{264}{32}$, $8\frac{1}{4}$, $\frac{.0264}{.032}$

2. A rectangular region whose adjacent sides have measures 3.2 and 6.7 is separated into 16 smaller congruent rectangular regions. What is the measure of each of the smaller regions?

3. a. Which of these numbers is the greater,

$$117 \div 36 \quad \text{or} \quad 113.4 \div 35?$$

   b. How much greater?

4. If a plane travels 2750 miles in 5.5 hours, how far will it travel in 3 hours?

EXTENDING THE DIVISION PROCESS

Exploration

Consider the quotient $18.7 \div 2.5$. We know that

$$18.7 \div 2.5 = \frac{18.7}{2.5} = \frac{187}{25}$$

We can find by the division process

$$\begin{array}{r} 7 \\ 25\,\overline{)\,187} \\ \underline{175} \\ 12 \end{array}$$

that $$\frac{187}{25} = 7\frac{12}{25}$$

We know that $$7\frac{12}{25} = 7\frac{48}{100} = 7.48$$

Now we know that $18.7 \div 2.5 = 7.48$, but the division process did not give us this name for the answer. The question is this: Can we extend the division process so that it gives us this decimal name?

To get an idea about the answer to this question, study the following quotients. Show that each answer is correct.

1. $18{,}700 \div 25 = \qquad 748$

2. $1870 \div 25 = \frac{1870}{25} = 74\frac{20}{25} = 74\frac{80}{100} = \qquad 74.8$

3. $187 \div 25 = \frac{187}{25} = 7\frac{12}{25} = 7\frac{48}{100} = \qquad 7.48$

4. $18.7 \div 25 = \frac{18.7}{25} = \frac{187}{250} = \frac{748}{1000} = \qquad .748$

5. What do you think is the decimal for

$$187{,}000 \div 25?$$

for $$1.87 \div 25?$$

How can you explain the fact that the answers have the same digits and differ only in the position of the decimal point? Consider exercises 1 and 2, to see how you can use the answer for exercise 1 to find the answer for exercise 2.

$$\frac{18{,}700}{25} = 748$$

$$\frac{1870}{25} = \frac{1870 \times 10}{25 \times 10}$$

$$= \frac{18{,}700 \times 1}{25 \times 10}$$

$$= \frac{18{,}700}{25} \times \frac{1}{10}$$

$$= 748 \times \frac{1}{10}$$

$$= 74.8$$

Now consider the division process for 18.7 ÷ 25 again. Think of 187 as 187.00 and divide as usual.

6. $18.7 \div 2.5 = \frac{187}{25}$.

a.
$$\begin{array}{r} 7. \\ 25\,\overline{)\,187.00} \\ \underline{175} \\ 12 \end{array}$$

The 7 is 7 ones, so put a decimal point after 7. The remainder is 12 ones.

b.
$$\begin{array}{r} 7.4 \\ 25\,\overline{)\,187.00} \\ \underline{175} \\ 12\;0 \end{array}$$

Think of 12 ones as 120 tenths. 120 ÷ 25 is about 4, ~~so~~ 120 tenths ÷ 25 is about 4 tenths. Write the 4 in tenths' place.

c.
$$\begin{array}{r} 7.4 \\ 25\,\overline{)\,187.00} \\ \underline{175} \\ 12\;0 \\ \underline{10\;0} \\ 2\;0 \end{array}$$

25 × 4 tenths is 100 tenths. The remainder is 20 tenths.

d.
$$\begin{array}{r} 7.48 \\ 25\,\overline{)\,187.00} \\ \underline{175} \\ 12\;0 \\ \underline{10\;0} \\ 2\;00 \\ \underline{2\;00} \end{array}$$

Think of 20 tenths as 200 hundredths.
25 × 8 hundredths = 200 hundredths
Write the 8 in hundredths' place.

$18.7 \div 2.5 = 7.48$

You see that you can find the digits in the answer just as though you were dividing 18700 by 25. If you place the decimal point carefully you do not need to think again of tenths and hundredths.

Look at another example:

7. Find the decimal for 87.4 ÷ 25.

a.
$$\begin{array}{r} 3. \\ 25\,\overline{)\,87.4} \\ \underline{75} \\ 12\ 4 \end{array}$$

The remainder is really 12.4

The division shows that

$$87.4 = (25 \times 3) + 12.4$$

Since $\frac{12.4}{25} = \frac{124}{250}$ or $\frac{124}{25} \times \frac{1}{10}$,

place the next digit in tenths' place.

b.
$$\begin{array}{r} 3.4 \\ 25\,\overline{)\,87.4} \\ \underline{75} \\ 12\ 4 \\ \underline{10\ 0} \\ 2\ 4 \end{array}$$

25 × 4 tenths = 100 tenths.

The remainder is really 2.4

Think of 2.4 as 2.40, or 240 hundredths, and continue the process.

c.
$$\begin{array}{r} 3.496 \\ 25\,\overline{)\,87.400} \\ \underline{75} \\ 12\ 4 \\ \underline{10\ 0} \\ 2\ 40 \\ \underline{2\ 25} \\ 150 \\ \underline{150} \end{array}$$

$$87.4 \div 25 = 3.496$$

8. How does this process compare with what you would have done to find the decimal numeral for 87,400 ÷ 25?

9. Exercise 7 shows that you can use this extended process without rewriting the problem so that both numerator and denominator of the fraction are whole numbers. Rename the fraction so the denominator is a whole number.

Look at this example:

Find the decimal for 12.68 ÷ .4

a. $12.68 \div .4 = \frac{12.68}{.4} = \frac{126.8}{4}$

Now proceed as before:

b.
$$\begin{array}{r} 31.7 \\ 4\,\overline{)\,126.8} \\ \underline{12} \\ 6 \\ \underline{4} \\ 2\,8 \\ \underline{\underline{2\,8}} \end{array}$$

$12.68 \div .4 = 31.7$

10. You know that $12.68 \div .4 = 31.7$ is a true sentence if $.4 \times 31.7 = 12.68$ is a true sentence. Use the multiplication process to show that this sentence is true.

11. Use multiplication to check the answers for exercises 6 and 7.

Exercise Set 19

1. Find decimal numerals for these quotients.

   a. $231 \div 15$

   b. $23.1 \div .15$

   c. $.231 \div 1.5$

   d. $2.31 \div 15$

   e. $5.85 \div 7.5$

   f. $5850 \div 7.5$

   g. $.585 \div 75$

   h. $58.5 \div .75$

2. Rename these numbers so the denominator of the fraction is a whole number.

   a. $\frac{73.6}{.25}$

   b. $\frac{.097}{3.265}$

   c. $\frac{38.92}{5.1}$

   d. $\frac{685}{8.2}$

   e. $\frac{390}{.127}$

   f. $\frac{.063}{1.85}$

   g. $\frac{649}{.36}$

   h. $\frac{1.267}{5.9}$

3. Find decimal names for these quotients. Check by multiplication.

   a. $1008 \div .6$

   b. $213.9 \div 3.75$

   c. $646 \div 6.8$

   d. $30.94 \div 2.6$

4. Use the sentence $18.7 \div 2.5 = 7.48$ to write the decimal numeral for each of these quotients:

a. $187 \div 25$ b. $18700 \div 250$ c. $18.7 \div .25$

5. Use $\frac{9}{25} = .36$ to write the decimal numeral for each of these.

a. $90 \div 25$ b. $9 \div 250$ c. $900 \div 25$

6. Make each sentence true by placing the decimal point and writing any needed zeros in the numeral for the second factor shown.

a. $2.63 \times 31 = 8.153$

b. $26.3 \times 31 = 815.3$

c. $.263 \times 31 = 8.153$

Express the answer to each of the following problems as a decimal numeral.

7. If 1 centimeter were exactly .4 inches, what would the measure of an inch be in centimeters?

8. A car used 10.5 gallons of gasoline in traveling 163.8 miles. How many miles did it travel per gallon of gasoline?

9. How long will it take to travel 144 miles at 32 miles per hour?

ESTIMATING RATIONAL NUMBERS USING DECIMALS

Exploration

Why is it that some rational numbers have no decimal names?

1. Consider these decimals. .7, 2.85, .037, 18.279.
   What is the denominator of the fraction form of each decimal?

   Since the decimal system has ten as its base, only 10, or 10^2, or 10^3, or some other power of 10 can be the denominator of the fraction form of a decimal.

2. Write the complete factorization of 10^1, of 10^2, of 10^3.
   What different numbers are prime factors of any power of 10?

3. Write the complete factorization for the denominator of each fraction.

   a. $\frac{5}{6}$ c. $\frac{10}{21}$ e. $\frac{11}{32}$

   b. $\frac{7}{15}$ d. $\frac{9}{40}$ f. $\frac{13}{36}$

4. Suppose each number named by a fraction in exercise 3 is to be renamed by another fraction with a whole number numerator and denominator. What prime factors must the denominator of the new fraction have to rename $\frac{5}{6}$? to rename each of the other numbers? Does 10 have these factors? Does 10^2? Does any power of 10?

5. Which of the numbers in exercise 3 have decimal names?

What happens if we try our extended division process on quotients which do not have decimal names?

6. Let's try $\frac{11}{9}$.

$$\frac{11}{9} = 11 \div 9$$

$$\begin{array}{r} 1 \\ 9\,\overline{)\,11} \\ \underline{9} \\ 2 \end{array} \qquad \frac{11}{9} = 1 + \frac{2}{9}$$

$$\begin{array}{r} 1.2 \\ 9\,\overline{)\,11.0} \\ \underline{9} \\ 2\;0 \\ \underline{1\;8} \\ 2 \end{array} \qquad \frac{11}{9} = 1.2 + \frac{.2}{9} = 1.2 + \frac{2}{90}$$

$$= 1.2 + \left(\frac{2}{9} \times \frac{1}{10}\right)$$

$$\begin{array}{r} 1.22 \\ 9\,\overline{)\,11.00} \\ \underline{9} \\ 2\;0 \\ \underline{1\;8} \\ 2 \end{array} \qquad \frac{11}{9} = 1.22 + \frac{.02}{9} = 1.22 + \frac{2}{900}$$

$$= 1.22 + \left(\frac{2}{9} \times \frac{1}{100}\right)$$

Suppose you divide 11.000 by 9. What do you think the quotient will be?

Is there any point in continuing the division process?

We have found that

$$\frac{11}{9} = 1.22 + \frac{.02}{9} = 1.22 + \frac{2}{900}$$

During the process we found that

$$\frac{11}{9} = \underset{\substack{\uparrow \\ \text{units digit in quotient}}}{1} + \frac{2}{9} \longleftarrow \text{first remainder}$$

$$\frac{11}{9} = 1.\underset{\substack{\uparrow \\ \text{tenths' digit in quotient}}}{2} + \frac{.2}{9} \longleftarrow \text{second remainder}$$

If we continued we would find

$$\frac{11}{9} = 1.22\underset{\substack{\uparrow \\ \text{thousandths' digit in quotient}}}{2} + \frac{.002}{9} \longleftarrow \text{fourth remainder}$$

What would the next step show?

Several more examples may help to show what can happen.

7. $5 \div 6 = \frac{5}{6}$

$$\begin{array}{r} .83 \\ 6\,\overline{)5.00} \\ \underline{4\,8} \\ 20 \\ \underline{18} \\ 2 \end{array}$$

$$\frac{5}{6} = .83 + \frac{.02}{6}$$

$$= .83 + \frac{2}{600}$$

The next step would show:

$$\frac{5}{6} = .833 + \frac{.002}{6}$$

$$= .833 + \frac{2}{6000}$$

8. $\frac{5}{11} = 5 \div 11$

$$\begin{array}{r} .454 \\ 11\,\overline{)\,5.000} \\ \underline{4\;4} \\ 60 \\ \underline{55} \\ 50 \\ \underline{44} \\ 6 \end{array}$$

We can stop now. Do you see why?

We have found

$$\frac{5}{11} = .4 + \frac{.6}{11}$$

$$\frac{5}{11} = .45 + \frac{.05}{11}$$

$$\frac{5}{11} = .454 + \frac{.006}{11}$$

Can you write what the next step would show if you continued the division process?

9. $\frac{1}{8} = 1 \div 8$

```
    .125
8 ) 1.000
    8
    20
    16
     40
     40
      0
```

$$\frac{1}{8} = .1 + \frac{.2}{8}$$

$$\frac{1}{8} = .12 + \frac{.04}{8}$$

$$\frac{1}{8} = .125$$

We can think of the division process for $\frac{1}{8}$ this way: At the first step we found $\frac{1}{8} > .1$ and $\frac{1}{8} < .2$. We might say that we have estimated $\frac{1}{8}$ in tenths.

At the second step we found an estimate for $\frac{1}{8}$ in hundredths.

$$\frac{1}{8} > .12 \quad \text{and} \quad \frac{1}{8} < .13$$

At the third step we found an estimate for $\frac{1}{8}$ in thousandths. As soon as we subtracted we found an exact numeral for $\frac{1}{8}$ in thousandths.

$$\frac{1}{8} = .125$$

10. Can you find estimates for $\frac{11}{9}$, $\frac{5}{6}$, and $\frac{5}{11}$ in ten thousandths?

Will the division process ever give a decimal numeral for any of these numbers?

11. a. What is the estimate for $\frac{2}{3}$ in ten-thousandths?
    b. Does $\frac{1}{16}$ have a decimal numeral?
    c. Does $\frac{1}{40}$ have a decimal numeral?
    d. Find a decimal for $\frac{8}{125}$ if there is one.

Exercise Set 20

1. Which of the following numbers have decimal names and which do not? If a number has a decimal name, find it by the division process.

a.	$\frac{5}{12}$	c.	$\frac{57}{200}$	e.	$\frac{27}{14}$
b.	$\frac{7}{18}$	d.	$\frac{31}{40}$	f.	$\frac{69}{25}$

2. Use the division process to estimate each number in tenths as shown in the example.

Example: $\frac{15}{7} = 15 \div 7$

$$\begin{array}{r} 2.1 \\ 7\;\overline{)\,15.0} \\ \underline{14} \\ 1\;0 \\ \underline{7} \\ 3 \end{array}$$

$\frac{15}{7}$ is between 2.1 and 2.2

a.	$\frac{19}{8}$	d.	$\frac{3}{8}$
b.	$\frac{25}{6}$	e.	$\frac{4}{11}$
c.	$\frac{131}{14}$	f.	$\frac{365}{7}$

3. Estimate each of these numbers to the nearest smaller or the exact thousandth as shown in the example.

Example: $\frac{1}{6}$

$$\begin{array}{r} .166 \\ 6\,\overline{)\,1.000} \\ \underline{6} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 4 \end{array}$$

Number	Nearest Smaller or Exact Thousandth	Number	Nearest Smaller or Exact Thousandth	Number	Nearest Smaller or Exact Thousandth
$\frac{1}{2}$	.500	$\frac{4}{5}$		$\frac{2}{7}$	
$\frac{1}{3}$		$\frac{4}{7}$		$\frac{2}{9}$	
$\frac{1}{4}$		$\frac{4}{9}$		$\frac{3}{4}$	
$\frac{1}{5}$		$\frac{5}{6}$		$\frac{3}{5}$	
$\frac{1}{6}$	.166	$\frac{5}{7}$		$\frac{3}{7}$	
$\frac{1}{7}$		$\frac{5}{8}$		$\frac{3}{8}$	
$\frac{1}{8}$		$\frac{5}{9}$		$\frac{6}{7}$	
$\frac{1}{9}$		$\frac{2}{3}$		$\frac{7}{8}$	
$\frac{1}{10}$		$\frac{2}{5}$		$\frac{7}{9}$	
				$\frac{8}{9}$	

4. Use the table you made for exercise 3 or the division process to answer these questions. Which is larger,

a. $\frac{3}{7}$ or $\frac{4}{9}$?

b. $\frac{5}{6}$ or $\frac{21}{25}$?

c. $\frac{5}{9}$ or $\frac{7}{13}$?

d. $\frac{5}{27}$ or $\frac{2}{11}$?

e. .6254 or $\frac{5}{8}$?

f. 1667 or $\frac{10^4}{6}$?

5. Express the following numbers as hundredths plus remainder as shown in the example.

Example: $55 \div 17$

$$\begin{array}{r}
3.23 \\
17\,\overline{)\,55.00} \\
\underline{51} \\
4\,0 \\
\underline{3\,4} \\
60 \\
\underline{51} \\
9
\end{array}$$

$$55 \div 17 = 3.23 + \frac{.09}{17}$$

a. $2\frac{1}{2} \div .6$

b. $\frac{5}{8} \div \frac{8}{5}$

c. $\frac{365}{12}$

d. $\frac{5280}{16}$

e. $7\frac{5}{16} \div \frac{3}{8}$

f. $\frac{238,000}{186,000}$

BRAINTWISTER

6. Which of these numbers have decimal names? If a number has a decimal name, find it by the process shown in the example.

Example: $\frac{7}{80}$

$$\frac{7}{80} = \frac{7}{5 \times 2^4} = \frac{7 \times 5^3}{5^4 \times 2^4} = \frac{7 \times 5^3}{10^4} = \frac{7 \times 125}{10^4} = \frac{875}{10^4} = .0875$$

a. $\frac{11}{40}$

d. $\frac{51}{800}$

b. $\frac{49}{36}$

e. $\frac{13}{90}$

c. $\frac{50}{28}$

f. $\frac{3}{32}$

Exercise Set 21

1. Complete the table below. The measurements given are for rectangular regions.

	Measure of One Side	Measure of Adjacent Side	Perimeter	Area of Region
a.	8 feet	3 feet	22 feet	24 square feet
b.		10 yards		200 sq. yards
c.	$3\frac{3}{4}$ inches			$28\frac{1}{8}$ sq. inches
d.	$25\frac{3}{4}$ miles	$4\frac{1}{3}$ miles		
e.	5 yards			$65\frac{5}{6}$ sq. yards
f.		25.6 feet		87.04 sq. feet
g.	3.6 Miles			201.6 sq. mi.
h.	7.6 inches	9.8 inches		

2. Pete made a scale drawing of the floor plan for a house. He used a length of $\frac{1}{2}$ inch to represent 5 feet. What were the lengths of the segments he drew to represent a room 18 feet by 30 feet?

3. ✓ Mrs. Brown mailed five boxes of cookies and four boxes of candy to friends at Christmas. Each box of cookies weighed $2\frac{1}{4}$ pounds and each box of candy weighed $\frac{3}{4}$ pounds. How many pounds of cookies and candy did she mail?

4. Bill is using a ruler he has made to measure lengths. He thinks it is a foot long, and he has very carefully marked it off in twelve parts of equal length. If Bill's ruler is really 13.08 inches in length, how long is one of Bill's inches?

5. ✓ When the Gray family left on a trip, the speedometer read 717.6 miles. At the end of the trip, the speedometer read 1202.1 miles.

   a. How many miles had the Grays traveled?

   b. If they made the trip in 8.5 hours, what was their average speed?

6. ✓ The circumference of the world at the equator is 24,902.37 miles. The circumference of the world at a meridian is 24,860.44 miles. How many miles greater is the circumference at the equator than at a meridian?

7. A salesman kept a record of the distances he traveled in one week. The distances recorded were: 76.4; 85.9; 75.3; 92.5; and 100.4 miles.

a. What was the total number of miles the salesman traveled during the week?

b. What was the average distance traveled per day?

c. If the car averaged 17.5 miles to the gallon, how many gallons of gasoline were used during the week?

d. At $ .30 per gallon, what was the cost of the gasoline?

e. Using the distance traveled during this one week as an average, find the total number of miles the salesman traveled in 50 weeks.

f. Using the information given in exercises c and d for average mileage and average price of gasoline, find the cost of gasoline used during a year.

8. The population of Chicago is $4\frac{1}{2}$ times that of Boston. The population of New York City is $2\frac{1}{10}$ times that of Chicago. The population of New York City is how many times as great as the population of Boston?

9.✓ Kent has a rectangular garden $9\frac{1}{4}$ feet wide and $12\frac{2}{3}$ feet long. He wants to put a wire fence along its four sides. The wire sells for $13\frac{1}{2}$ cents a foot. How much will Kent have to pay for enough wire for his garden?

10.✓ Bill ran 100 yards in 15.6 seconds. Ray ran 75 yards in 11.8 seconds. Who ran faster?

11.✓ A very fast runner can run 100 yards in 9.4 seconds. Express this rate in miles per hour.

12. The moon is about 238,000 miles from the earth. The sun is about 93,000,000 miles from the earth.

    a. With the distance to the moon as a unit, what is the measure of the distance to the sun?

    b. If a space ship could reach the moon in 18 hours, at that rate how many days would it take for a spaceship to reach the sun from the earth?

13. The populations of the 5 largest cities in the United States are listed below. What part of the 180,000,000 people in the United States live in all of these cities?

New York	10,700,000
Los Angeles	6,700,000
Chicago	6,200,000
Philadelphia	4,300,000
Detroit	3,800,000

✓14. Suppose that the average cost of driving a car on a free road is 7 cents per mile. The distance from Philadelphia to Pittsburgh is 300 miles along the Pennsylvania turnpike and 360 by a free road. Suppose the toll on the turnpike is $3.50. Which is the cheaper way to travel from Philadelphia to Pittsburgh, and by how much?

15. BRAINTWISTER

The measure of 1 inch in centimeters is about 2.54. Use this to complete the following table. Estimate measures which do not have decimal names to the nearest thousandth.

a. The measure of 1 foot in centimeters.

b. The measure of 1 foot in meters.
(1 meter = 100 centimeters)

c. The measure of 1 yard in meters.

d. The measure of 1 mile in kilometers (1000 meters)

e. The measure of 1 centimeter in inches.

f. The measure of 1 meter in inches.

g. The measure of 1 kilometer in miles.

Exercise Set 22

1. Find a decimal name for each sum.

a. 356 + 47.8
b. 45 + 17.17
c. 0.89 + 0.75
d. 59.62 + 3.84
e. 373.4 + 75.99
f. 0.4 + 0.6

2. Find decimal names for these numbers.

a. 56 - 9.3
b. 923.1 - 74.8
c. 57.48 - 36.92
d. 74 - 22.45
e. 37.15 - 29.8
f. 469.1 - 89.74

3. Multiply:

a. 0.4 by 0.2
b. 38.9 by 2.67
c. 760 by 4.58
d. 6846 by 5.3
e. 347.8 by 5.6
f. 7.92 by 8.9

4. Find a decimal for each quotient.

a. 1144 ÷ 2.3
b. 36.75 ÷ 0.49
c. 142.5 ÷ 0.57
d. 222.7 ÷ 6.9
e. 3630 ÷ 0.30
f. 29.37 ÷ 8.9

5. Perform the following operations:

a. 54.72 ÷ 5.7
b. 56 - 9.3
c. 29.9 + 63
d. 220.5 ÷ 0.65
e. 37.9 × 4.2
f. 87.4 - 39.56
g. 68.08 ÷ 0.92
h. 34 + 7.69
i. 7.60 × 13.5
j. 5780 ÷ 8.5
k. 7.65 - 3.6
l. 979.4 + 32.85
m. 388 × 0.74
n. 227.7 ÷ 6.9

6. Perform the following operations:

a. 87.4 - 39.56
b. 68.08 ÷ 0.92
c. $17\frac{5}{12} - 9\frac{1}{2}$
d. $2\frac{1}{2} \div 3\frac{5}{8}$
e. 543.9 + 74.35
f. 388 × 0.74
g. 227.7 ÷ 6.9
h. $7\frac{1}{2} \times 3\frac{5}{6}$
i. $44\frac{2}{3} + 7\frac{5}{9}$
j. 0.3 × 0.2

7. Perform the following operations:

a. 30.66 ÷ 4.2
b. $92\frac{1}{4} + 73\frac{5}{6}$
c. 59 ÷ 0.78
d. $80 - 7\frac{2}{9}$
e. 25.6 × 0.98
f. $4\frac{1}{3} \times 6\frac{2}{7}$
g. $5\frac{2}{3} \div \frac{4}{5}$
h. 31 - 5.5
i. $17\frac{1}{2} \div 2\frac{1}{3}$
j. 481.32 ÷ 0.84

5.1	7.2	0.3	2.4	4.5
6.9	1.5	2.1	4.2	4.8
1.2	1.8	3.9	6.0	6.6
3.0	3.6	5.7	6.3	0.9
3.3	5.4	7.5	0.6	2.7

8. Copy the square above.

   a. Add the numbers named by the decimals in each column and record the sum for each column.

   b. Add the numbers named by the decimals in each row and record the sum for each row.

   c. Begin at the lower left-hand corner and add diagonally. Record the sum.

   d. Begin in the upper left-hand corner. Add diagonally. Record the sum.

   e. Is each sum the same rational number? What is the number?

   f. Is the square a magic square?

Practice Exercises

I. Find the number that t represents.

a) $5\frac{2}{3} + 1\frac{1}{9} = t$

b) $t \times 64 = 14848$

c) $3\frac{1}{4} \times 2\frac{1}{3} = t$

d) $^{-}625 + t = {}^{-}480$

e) $7236 \div t = 67$

f) $8\frac{1}{3} = \frac{t}{12}$

g) $^{+}4378 + {}^{-}3487 = t$

h) $19 \div \frac{5}{6} = t$

i) $^{-}8483 + t = {}^{-}3479$

j) $962.56 \div 6.4 = t$

k) $28\frac{9}{16} - t = 12\frac{1}{4}$

l) $3\frac{1}{2} \div 1\frac{1}{3} = t$

m) $^{+}2740 + t = {}^{+}1769$

n) $27\frac{1}{2} \times 5\frac{5}{9} = t$

o) $684.84 = 3.9 \times t$

p) $7\frac{1}{4} + 19\frac{2}{3} + 11\frac{5}{6} = t$

q) $3\frac{4}{7} + t + 3\frac{1}{3} = 12\frac{4}{12}$

r) $27.45 + t = 78$

s) $\frac{125}{1000} = \frac{t}{8}$

t) $4\frac{5}{6} \times \frac{3}{8} = t$

II. Solve

a) $\frac{5}{16} \div 4$

b) $48.90 \div 30$

c) $5 \times \frac{5}{6}$

d) $15.789 + 13.763$

e) $5\frac{1}{3} - 1\frac{3}{7}$

f) $\frac{7}{8} \times 36$

g) $38.400 \div 60$

h) $4\frac{1}{2} \div 1\frac{1}{2}$

i) $38.050 \div 125$

j) $8\frac{3}{5} \times 4$

k) $428.07 \div .57$

l) $931.44 - 265\ 9$

m) $3\frac{1}{12} \times 2\frac{2}{5}$

n) $9 \div 4\frac{2}{3}$

o) $577.28 \div 6.4$

p) $4\frac{1}{6} \div \frac{3}{4}$

q) 162.4×87.5

r) $468.394 - 288.54$

s) $3\frac{1}{2} + 4\frac{3}{8} + 2\frac{1}{3}$

t) $\frac{5}{8} \div 2\frac{1}{2}$

III. Add:

1. $\begin{array}{r}1463\\423\\4684\\\underline{5736}\end{array}$
2. $\begin{array}{r}69053\\2928\\75\\\underline{71089}\end{array}$
3. $\begin{array}{r}2\frac{1}{2}\\\frac{5}{6}\\4\frac{2}{3}\\\underline{1\frac{1}{4}}\end{array}$
4. $\begin{array}{r}843{,}695\\24{,}763\\927{,}616\\\underline{44{,}464}\end{array}$
5. $\begin{array}{r}2\frac{1}{3}\\6\frac{5}{6}\\3\frac{1}{2}\\\underline{\frac{3}{4}}\end{array}$

Subtract:

1. $\begin{array}{r}8902\\\underline{4723}\end{array}$
2. $\begin{array}{r}39668\\\underline{25756}\end{array}$
3. $\begin{array}{r}27\frac{1}{12}\\\underline{8\frac{2}{3}}\end{array}$
4. $\begin{array}{r}658.374\\\underline{167.41}\end{array}$
5. $\begin{array}{r}2\frac{4}{5}\\\underline{\frac{3}{4}}\end{array}$

Multiply:

1. $\begin{array}{r}365\\\underline{427}\end{array}$
2. $\begin{array}{r}87.91\\\underline{2.8}\end{array}$
3. $\begin{array}{r}8\frac{5}{8}\\\underline{4\frac{2}{3}}\end{array}$
4. $\begin{array}{r}3846\\\underline{508}\end{array}$
5. $\begin{array}{r}\$4.98\\\underline{36}\end{array}$

Divide:

1. $35\overline{)147{,}535}$
2. $24\overline{)5{,}784}$
3. $72\overline{)14.544}$
4. $4\frac{2}{3} \div 1\frac{1}{6}$
5. $6 \div \frac{5}{6}$

Braintwister

Egyptian Fractions

The ancient Egyptians expressed all of their fractions as unit fractions, such as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$ etc. (One exception was $\frac{2}{3}$) Instead of writing $\frac{7}{12}$ they had to write $\frac{1}{2} + \frac{1}{12}$, or $\frac{1}{4} + \frac{1}{4} + \frac{1}{12}$.

Express each fraction as the sum of two unit fractions.

a) $\frac{9}{16}$ c) $\frac{5}{8}$ e) $\frac{1}{3}$ g) $\frac{3}{8}$

b) $\frac{5}{12}$ d) $\frac{5}{6}$ f) $\frac{3}{4}$ h) $\frac{9}{20}$

Review

SET I

Part A

1. Choose an integer from Set A to complete and make each of the following true statements.

   Set A = $\{0, {}^{-}1, {}^{-}2, {}^{-}3, {}^{-}4 \ldots\}$

   a) $^{+}3 + \underline{\quad} = {}^{+}2$

   b) $\underline{\quad} + {}^{-}2 = {}^{-}9$

   c) $\underline{\quad} + {}^{+}3 = {}^{-}6$

   d) $^{+}10 + \underline{\quad} = {}^{-}3$

   e) $^{+}5 + \underline{\quad} = {}^{-}12$

   f) $\underline{\quad} + {}^{+}15 = {}^{-}15$

   g) $^{+}9 + \underline{\quad} = 0$

   h) $^{+}18 + \underline{\quad} = {}^{+}9$

2. Choose an integer from Set B to complete and make each of the following true statements.

   Set B = $\{0, {}^{+}1, {}^{+}2, {}^{+}3, {}^{+}4 \ldots\}$

   a) $^{-}4 + \underline{\quad} = {}^{+}2$

   b) $^{+}5 + \underline{\quad} = {}^{+}16$

   c) $^{+}9 + \underline{\quad} = {}^{+}12$

   d) $\underline{\quad} + {}^{-}6 = {}^{+}2$

   e) $^{-}7 + \underline{\quad} = {}^{-}7$

   f) $\underline{\quad} + {}^{-}18 = {}^{-}6$

   g) $\underline{\quad} + {}^{-}12 = {}^{+}6$

   h) $^{+}3 + \underline{\quad} = {}^{+}51$

   i) The intersection of Sets A and B is Set C. Name the members of Set C.

3. Would the arrow drawn for each of the following unknown addends be named by a positive or negative integer?

   a) $^{+}2 + n = {}^{+}8$

   b) $n + {}^{-}6 = 0$

   c) $^{+}5 + n = {}^{+}1$

   d) $n + {}^{+}3 = {}^{-}9$

   e) $^{+}16 + n = {}^{+}32$

   f) $n + {}^{+}17 = {}^{-}6$

   g) $^{-}8 + n = {}^{+}8$

   h) $^{+}4 + n = {}^{+}3$

   i) $^{-}15 + n = {}^{+}9$

   j) $n + {}^{-}1 = {}^{+}21$

4. Rename each fraction so that it may be written as a decimal. Complete the chart.

Fraction	Multiply By	New Fraction	Decimal
a) $\frac{1}{5}$	$\frac{2}{2}$ $\frac{1 \times 2}{5 \times 2}$	$\frac{2}{10}$	.2
b) $\frac{4}{25}$			
c) $\frac{3}{2}$			
d) $\frac{1}{4}$			
e) $\frac{18}{50}$			
f) $\frac{7}{20}$			
g) $\frac{3}{4}$			
h) $\frac{5}{8}$			

5. Find the number that n represents. Use decimal or fraction form for your work. Write the answer as a decimal.

a) $.84 \times 6.8 = n$
b) $9 \div .3 = n$
c) $25 \times .25 = n$
d) $45.56 \div 68 = n$
e) $1.628 \div 4.4 = n$
f) $.0256 \div 1.6 = n$
g) $.25 \times .25 = n$
h) $.25 \div 4 = n$

6. Write the set of numbers named by the reciprocals of the members of Set R, call it Set S.

$R = \{1, \frac{1}{2}, \frac{2}{4}, \frac{4}{8}, \frac{6}{12}, \frac{8}{16}, \frac{10}{20}, \frac{12}{24}\}$

7. Complete the following to make them true sentences.

a) ${}^{-}23 + \_\_\_\_ = {}^{-}41$

b) $\frac{3}{4} \div \frac{2}{3} = \_\_\_\_$

c) $6\frac{2}{3} \times \_\_\_\_ = 58\frac{1}{3}$

d) ${}^{-}14 + \_\_\_\_ = {}^{+}37$

e) $1\frac{1}{2} \times \frac{5}{6} = \_\_\_\_$

f) $\frac{3}{4} \div \_\_\_\_ = \frac{1}{6}$

g) $\frac{2}{3} \times \_\_\_\_ = \frac{5}{6}$

h) $\_\_\_\_ + {}^{-}8 = {}^{-}32$

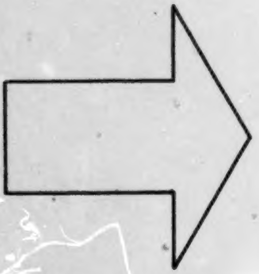
i) ${}^{+}45 + {}^{-}17 = \_\_\_\_$

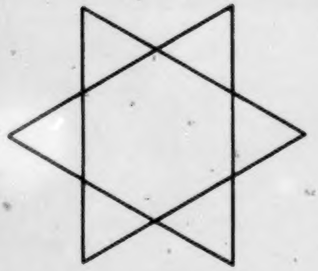
j) $\frac{5}{8} \div 1\frac{1}{4} = \_\_\_\_$

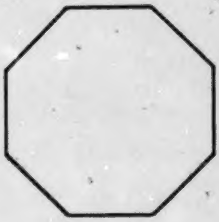
8. Complete the chart below.

Set of Integers	Largest Integer	Smallest Integer
a) ${}^{-}26, {}^{+}14, {}^{+}26, {}^{-}8, {}^{+}40$		
b) ${}^{+}13, {}^{-}9, {}^{+}16, {}^{+}19, {}^{-}8$		
c) ${}^{+}4, {}^{-}3, {}^{+}7, 0, + 1$		
d) ${}^{-}46, {}^{-}28, {}^{++}2, {}^{-}1, {}^{+}279$		
e) ${}^{+}7, {}^{+}3, {}^{+}5, 0, {}^{+}2$		

9. Copy these symmetric figures. Draw dotted lines to indicate as many axes of symmetry as you can of each figure.







10. Answer yes or no to these questions.

   a) Is there a smallest negative integer?
   b) Is there a largest negative integer?
   c) Is there a smallest positive integer?
   d) Is there a largest positive integer?
   e) Is it possible for two fractions to name the same rational number?
   f) Is it possible for a decimal to be named by a fraction?
   g) Is the set of negative integers closed under addition?
   h) Is the set of negative integers closed under subtraction?
   i) Is the set of positive integers closed under addition?
   j) Is the set of positive integers closed under subtraction?
   k). Is it possible for two fractions to name the same number when they have different denominators?

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. Mrs. Newlund baked two strawberry pies for dessert. She wants to serve each person $\frac{1}{3}$ of a pie. How many people can she serve with 2 pies?

2. In 1960 the population of Oak Harbor was 2,903. It is estimated that the population will be 2,750 in 1970. What will be the estimated loss in population during these ten years?

3. The Empire State Building is 1,472 ft. high. Without its T.V tower it is 1,250 ft. high. How tall is its T.V. tower?

4. Joe arrived at the bus station at 9:32 a.m. His bus is leaving at 10:05 a.m. How long will Joe have to wait for his bus?

5. During a game of "Ring Toss" Jane's ring fell 24 inches on one side of the peg. Dick's ring fell 19 inches on the opposite side of the peg. How far apart were their rings?

6. The water in a swimming pool weighs 148,808 pounds. Water weighs 8.36 pounds each gallon. How many gallons of water are in the pool?

7. On his route Ralph delivers 84 papers each day and he gets paid $2\frac{1}{4}$ cents for each delivery. How much does he earn in one week?

8. Tractor fuel costs 27.9 cents a gallon and one farmer used 2,430 gallons of fuel this year. How much did he pay for tractor fuel?

9. In the problem above, the farmer receives a tax rebate of 6 cents a gallon on the fuel. What is the final cost of the tractor fuel?

Review

SET II

Part A

1. Rewrite these sentences using letters to represent the numbers. Examples a and b give you some possible answers.

a) $2 \times 2 = 4$ $a \times a = d$

b) $3 \times 4 = 12$ $a \times b = c$

c) $3 - 0 = 3$

d) $^{+}4 + {}^{-}4 = 0$

e) $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

f) $\frac{3}{6} \times \frac{1}{2} = \frac{3}{12}$

g) $6 \div \frac{3}{5} = 3$

h) $2 \times 4 \times 9 = 72$

i) $12 \div 6 = 2$

j) $4.1 + 8.1 = 12.2$

k) $\frac{9}{12} \div \frac{1}{2} = 1\frac{1}{2}$

l) $5 \times 5 \times 5 = 5^3$

2. Write the symbol = or ≠ that makes each of the following a true sentence.

a) $.12$ ___ $\frac{3}{25}$

b) \$259.60 ÷ 41 ___ \$6.35

c) $3\frac{2}{7} \div \frac{6}{7}$ ___ $3\frac{5}{6}$

d) $5\frac{1}{2}$ ___ $\frac{44}{8}$

e) $\frac{45}{100}$ ___ $\frac{7}{20}$

f) 20.9×72 ___ 1540.8

g) $^{-}184 + {}^{-}276$ ___ $^{-}460$

h) $2\frac{1}{4}$ ___ $\frac{225}{1000}$

i) $^{-}38 - {}^{+}106$ ___ $^{-}144$

j) $\frac{3}{8}$ ___ $.375$

3. Rewrite the following as subtraction sentences and solve. Example a is done for you.

a) $^{-}4 + n = {}^{-}5$, $n = {}^{-}5 - {}^{-}4$, $n = {}^{-}1$

b) $^{+}6 + n = {}^{-}2$

c) $n + {}^{-}9 = {}^{-}12$

d) $^{+}16 + n = {}^{+}10$

e) $^{-}8 + n = {}^{+}3$

f) $n + {}^{+}21 = {}^{+}7$

g) $n + 0 = {}^{-}6$

4. Write the next six members of each set.

a) $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots\}$

b) $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$

c) $\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \ldots\}$

d) $\{1, \frac{1}{1}, \frac{2}{2}, \ldots\}$

e) $\{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \ldots\}$

5. Tell whether the following statements are true or false.

a) $^{+}4 + {}^{-}7 = {}^{-}2 + {}^{-}1$

b) $^{+}3 + {}^{+}6 = {}^{-}6 + {}^{-}3$

c) $\frac{1}{2} + \frac{3}{4} < 1 + \frac{1}{2}$

d) $\frac{2}{3} \times \frac{3}{2} > \frac{1}{2} \times \frac{2}{1}$

e) $^{+}5 - {}^{-}2 = {}^{-}5 - {}^{+}2$

f) $\frac{4}{3} - \frac{2}{3} < \frac{8}{6} - \frac{4}{6}$

g) $^{-}6 + {}^{+}7 = {}^{+}12 + {}^{-}11$

h) $\frac{3}{6} \times \frac{2}{4} = \frac{1}{4}$

i) $\frac{3}{16} \div \frac{2}{8} = \frac{9}{6} \quad \frac{6}{3}$

j) $^{-}3 + {}^{-}4 > 0$

6. Choose another name for the number from the row.

Example: $\frac{1}{10}$ a) 1. b) $\frac{10}{1}$ c) [✓ .1] d) .01

$\frac{26}{100}$ a) 2.6 b) $\frac{260}{10}$ c) 26.0 d) .26

$3\frac{5}{10}$ a) 0.35 b) 3.5 c) $\frac{350}{10}$ d) $\frac{35}{100}$

$12\frac{2}{100}$ a) 12.5 b) $\frac{122}{10}$ c) 12.02 d) $\frac{122}{100}$

$\frac{13}{1000}$ a) .130 b) .013 c) $\frac{1300}{10}$ d) 1.300

7. Rename each decimal so that it may be written as a fraction. Complete the chart.

Decimal	Fraction Name	Divide by	Fraction (Simplest form)
a) .5	$\frac{5}{10}$	$\frac{5}{5}$	$\frac{5 \div 5}{10 \div 5}$ $\frac{1}{2}$
b) .15			
c) .450			
d) .75			
e) .05			
f) .08			
g) .4			
h) .125			

8. Each of the following is true by one of the properties of rational numbers. Use the first letter of each word to identify the property. Example: D P A for distributive property of addition.

a) $\frac{3}{4} \times \frac{5}{6} = \frac{5}{6} \times \frac{3}{4}$

b) $\frac{1}{2} + (\frac{1}{4} + \frac{1}{5}) = (\frac{1}{2} + \frac{1}{4}) + \frac{1}{5}$

c) $3 \times 2\frac{1}{2} = (3 \times 2) + (3 \times \frac{1}{2})$

d) $\frac{1}{3} \times (3 \times 6) = (\frac{1}{3} \times 3) \times 6$

e) $4\frac{3}{4} \times 4\frac{1}{2} = (4 \times 4) + (\frac{3}{4} \times \frac{1}{2})$

f) $\frac{4}{5} + \frac{2}{3} = \frac{2}{3} + \frac{4}{5}$

g) $1 \times \frac{7}{8} = \frac{7}{8}$

h) $8\frac{2}{3} \times \frac{1}{4} = (8 \times \frac{1}{4}) + (\frac{2}{3} \times \frac{1}{4})$

i) $\frac{9}{10} \times 1 = \frac{9}{10}$

j) $.75 \times 2.5 = 2.5 \times .75$

9. Perform the operations in Column 1 and match with the correct result in Column 2.

	Column I	Column II
a)	$^{-}3,682 + ^{+}1,463$	____ $2\frac{4}{15}$
b)	$6\frac{1}{5} \div 1\frac{2}{3}$	____ $\frac{20}{29}$
c)	$2\frac{1}{4} \times \frac{7}{8}$	____ $44\frac{1}{24}$
d)	$\frac{136}{32}$	____ $3\frac{18}{25}$
e)	$17\frac{3}{8} + 26\frac{2}{3}$	____ $1\frac{31}{32}$
f)	$2\frac{1}{2} \div 3\frac{5}{8}$	____ $^{-}2,219$
g)	$5\frac{3}{5} - 3\frac{1}{3}$	____ 4.25
h)	$\frac{127}{25}$	____ 5.08

10. Complete these statements with the word needed to make them true.

a) The product of two rational numbers is always a __________ number.

b) A fraction with a denominator of 10 , 100 , 1,000 , ... may be written as a __________.

c) To write the reciprocal of a fraction, we _______ the fraction.

d) The product of a fraction and its __________ is 1.

e) One divided by a fraction is the __________ of the fraction.

f. To divide by a fraction __________ by its reciprocal.

11. Graph the following ordered pairs:

A	$(^{+}5, 0)$	G	$(^{+}5, ^{+}3)$	M	$(^{-}8, ^{+}3)$
B	$(^{+}10, ^{+}3)$	H	$(^{+}4, ^{+}3)$	N	$(^{-}7, 0)$
C	$(^{+}9, ^{+}4)$	I	$(^{+}4, ^{+}4)$	P	$(^{-}9, ^{+}3)$
D	$(^{+}5, ^{+}15)$	J	$(^{+}1, ^{+}11)$	Q	$(^{-}5, ^{+}3)$
E	$(^{+}4, ^{+}16)$	K	$(^{+}1, ^{+}16)$	R	$(^{+}1, ^{+}3)$
F	$(^{+}5, ^{+}4)$	L	$(^{-}8, ^{+}4)$	S	$(^{+}1, ^{+}4)$

Draw $\overline{AB}$, $\overline{QB}$, $\overline{PN}$, $\overline{CB}$, $\overline{CD}$, $\overline{DE}$, $\overline{DG}$, $\overline{FC}$, $\overline{EH}$, $\overline{IL}$, $\overline{EQ}$, $\overline{KM}$, $\overline{KJ}$, and $\overline{RS}$.

Graph the reflection on the x-axis and you have a calm day scene.

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. The speed of sound is generally 1088 feet per second. What is the speed of sound in miles per hour?

2. A plane flying at the speed of sound has a speed of "Mach I". How far will a plane flying at "Mach I" travel in thirty minutes?

3. A fisherman cast his bait 120 ft. upstream. He reeled it in after it had floated 75 ft. downstream. How far did he have to reel it in?

4. On Miss Meek's map 1 inch represented 12 miles. The distance from St. Louis to Bellville was $3\frac{3}{4}$ inches. What is the distance in miles from St. Louis to Bellville?

5. Mary went into business. The first month she lost $400.00. The second month she made a profit of $273.00. At the end of the second month has Mary made a loss or a profit? How much?

6. Octavius, an ancient Roman, was born in 75 B.C. He died in 37 B.C. How old was he when he died?

7. If you can cut twelve slices from one watermelon, how many slices can you cut from $2\frac{1}{4}$ watermelons?

8. The earth weight of a moon probe is 822 pounds. The moon weight is one-sixth that of the earth. What is the weight of the probe on the moon?

Review

SET III

Part A

1. Which of these are names for $^{+}3$.

   $^{+}5 + {}^{-}2$, $^{-}1 + {}^{-}2$, $^{+}8 - {}^{+}5$, $^{-}1 + {}^{+}4$

   Which of these are names for $^{-}1$?

   $^{+}7 + {}^{+}6$, $^{+}45 + {}^{-}46$, $^{-}146 + {}^{+}145$, $^{+}27 + {}^{-}26$

   Which of these are names for $^{-}82$?

   $^{-}1462 + {}^{+}1380$, $^{+}749 + {}^{-}832$, $^{+}869 + {}^{-}951$

   Which of these are names for 0 ?

   $^{+}8 + 0$, $^{+}42 + {}^{-}42$, $^{-}16 + 0$, $^{-}83 + {}^{+}83$

2. Using the symbol $<$, $=$, or $>$ make each of the following a true sentence.

   a) $^{-}27 + {}^{+}8$ ____ $^{+}27 + {}^{-}8$
   b) $^{+}293 + {}^{-}314$ ____ $^{+}319 + {}^{-}340$
   c) $^{+}37 + 0$ ____ $^{+}216 + {}^{-}216$
   d) $^{-}571 + {}^{+}589$ ____ $^{-}246 + {}^{+}264$
   e) $^{+}724 + {}^{-}837$ ____ $^{+}837 + {}^{-}724$
   f) $^{+}764 + {}^{-}961$ ____ $^{-}103 + {}^{-}94$
   g) $^{+}26 + {}^{+}85$ ____ $^{+}193 + {}^{-}294$
   h) $^{-}283 + {}^{+}13$ ____ $^{-}271 + 0$
   i) $^{-}47 + {}^{-}68$ ____ $^{+}808 + {}^{-}115$
   j) $^{+}709 + {}^{-}698$ ____ $^{-}698 + {}^{+}709$

3. R = {The set of rational numbers}

In which subset of Set R will the number that n represents be?

Example a is done for you.

a) $\frac{1}{2} + n = \frac{3}{4}$ {fractions}

b) $.02 \times 6.1 = n$

c) $6 \times n = 0$

d) $^{+}5 + n = {}^{+}2$ integers

e) $342 - 342 = n$

f) $16 \div \frac{3}{4} = n$

g) $4 - 6 = n$

h) $1 + n = 2$

4. Solve the following both as decimals and as fractions. An example is shown.

Example: $\frac{3}{4} \div \frac{1}{2}$, $.75 \div .5 = 1.5$, $\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \times \frac{2}{1} = \frac{6}{4} = 1\frac{1}{2}$

a) $\frac{1}{5} + \frac{1}{2}$ $\frac{7}{10}$) or (.7)

b) $3\frac{1}{4} - \frac{3}{4}$

c) $.15 \times 1.4$

d) $7.2 \div 9$

e) $.4 + 2.5 + .7$ 3.6 or $3\frac{6}{10}$ $\frac{3}{5}$

f) $7.25 \div .25$

g) $2\frac{2}{5} \times \frac{3}{10}$

h) $3.6 - .35$

i) $7\frac{1}{2} \div \frac{3}{10}$

j) $5\frac{4}{5} \times 1\frac{1}{2}$

X5. Use the words closed or not closed to complete and make these true sentences.

a) The set of rational numbers is __________ under the operation of multiplication.

b) The set of rational numbers greater than zero is __________ under the operation of subtraction.

c) The set of rational numbers is __________ under the operation of addition.

d) The set of rational numbers greater than zero is __________ under the operation of division.

6. Find n in each of the following.

a) $4 \times n = {}^-28$

b) $n + 1\frac{4}{7} = 3\frac{9}{14}$

c) ${}^+168 + n = {}^-312$

d) $\frac{5}{8} \div 3\frac{1}{3} = n$

e) $17.4 \times n = 400.2$

f) ${}^+94 + {}^-36 = n + {}^-214$

g) $129.92 \times n = 44.8 \times 2.9$

h) $1\frac{3}{4} + n = 6\frac{2}{3}$

i) $27.38 + 12.62 = 40 + n$

j) $5\frac{1}{2} \times 2\frac{1}{3} = n + 6\frac{1}{2}$

7. Graph this set of points $S = [A({}^-4, {}^+4), B({}^+6, {}^+4)$ $C({}^+2, {}^+4), D({}^-2, {}^+4) \ldots]$ Draw $\overline{AD}$. Extend $\overline{AD}$. The points of line AD seem equidistant to all points on which axis?

$T = [F({}^-3, {}^+6), G({}^-3, {}^-6), H({}^-3, {}^+2), J({}^-3, {}^-1) \ldots]$

Graph Set T. Draw $\overline{FG}$. Extend $\overline{FG}$. The points of line FG seem equidistant to all points of which axis?

R = [The set of points with y-coordinate 8)

W = [The set of points with x-coordinate 6)

Which set of points is equidistant to the y-axis? x-axis?

8. Which of the following have fraction names with the same numerator?

a) 6.23

b) 45.

c) 1.7

d) 623

e) 0.45

f) .17

g) .045

h) 62.3

i) 4.5

j) 17

9. Which of the numbers above have fraction names with the same denominator?

10. Are any two of the numerals above names for the same number?

11. Use the correct sign of operation to make true number sentences. Example a is worked.

a) $54 \underline{\ \div\ } 6 \underline{\ +\ } 8 \underline{\ =\ } 17$

b) $2\frac{1}{2} \underline{\qquad} 1\frac{1}{4} \underline{\qquad} 1\frac{1}{4} \underline{\qquad} 2\frac{1}{2}$

c) $^{+}19 \underline{\qquad} {}^{+}14 \underline{\qquad} {}^{-}28 \underline{\qquad} {}^{+}33$

d) $7 \underline{\qquad} 4 \underline{\qquad} 2 \underline{\qquad} 14$

e) $\frac{2}{3} \underline{\qquad} \frac{1}{2} \underline{\qquad} \frac{1}{3} \underline{\qquad} \frac{1}{1}$

f) $^{-}22 \underline{\qquad} {}^{+}16 \underline{\qquad} {}^{-}6 \underline{\qquad} 0$

g) $\frac{3}{4} \underline{\qquad} \frac{1}{3} \underline{\qquad} \frac{4}{2} \underline{\qquad} \frac{9}{2}$

h) $\frac{5}{8} \underline{\qquad} \frac{1}{4} \underline{\qquad} \frac{3}{4} \underline{\qquad} \frac{9}{8}$

12. What would be the decimal name of the last place to the right needed in:

Example a) The product of tens and tenths? ones

b) The product of tenths and tenths?

c) The product of hundreds and tenths?

d) The product of hundreds and hundredths?

e) The product of hundredths and tenths?

f) The product of thousands and thousandths?

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. Mr. Hank's car went 244.8 miles on 17 gallons of gasoline. At this rate, how far would he go on 12.5 gallons?

2. While baby-sitting for a neighbor, Marge receives 45 cents an hour. Last year she was paid $32.40. How many hours did she spend baby-sitting?

3. The area of the state of Oregon is 96,981 square miles. West Virginia is about $\frac{1}{4}$ the size of Oregon. What is the approximate area of West Virginia?

4. The area of West Virginia was found to be 24,181 square miles. How much too large was the approximate area found in Problem 3 above?

5. A farmer used 66.5 bushels of bean seed. It takes 1.75 bushels of seed to plant an acre. How many acres of beans did he plant?

6. The Taylor Oil Co. drilled an oil well. They struck oil at a depth of 5728 feet. The oil gushed from the well to a height of 169 feet. How far did the oil "gush" altogether?

7. The scale of miles on one map uses one inch to represent 160 miles. The air miles between two cities is 3,000 miles. What is the scale measurement in inches?

8. John can walk 2 miles in 25 minutes. One morning he walked $\frac{3}{5}$ of this distance in $\frac{1}{2}$ the time. How far did he walk and how long did it take him?

Individual Projects

1. Obtain a micrometer and learn to use it. A micrometer is an instrument that can make a measurement to the nearest thousandth of an inch. Demonstrate its use to your class.

2. Scientific notation is a name given to a quick and easy way to represent either very large or very small numbers. Find an explanation of this notation. Write some very large or very small numbers in scientific notation for your class.

3. Many great men have made important contributions to mathematics. Make a report about one of these famous mathematicians and his contribution.

4. There are some mathematical problems that are still mysteries to mathematicians. There are some about prime numbers, odd numbers and construction using only compass and straightedge. Look up one of these problems and try to solve it.

Group Activity

"Travel"

The object of the game is to "travel" as far as possible by being faster with the correct response. The first player stands beside the desk of the next player. A problem is given orally or a card flashed. The child giving the correct response first moves on to stand beside the next player's desk. For each turn the loser remains at the seat where he lost.

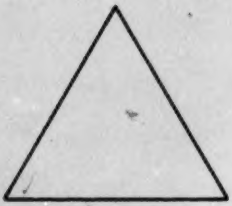
Chapter 7

VOLUME

SPACE FIGURES AND SPACE REGIONS

Any set of points in a plane is called a plane figure. Thinking in the same way, we shall call any set of points in space a space figure. The set of all points on a sphere is an example of a space figure. The surface of a box is a model of another space figure.

Which of the following are pictures of plane figures? Which ones are pictures of space figures?



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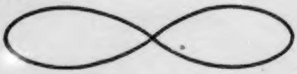

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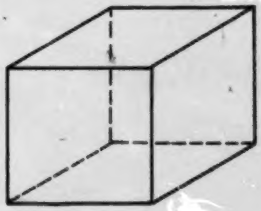

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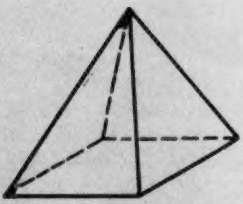

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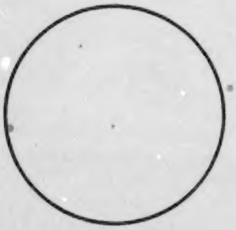

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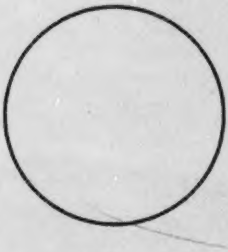

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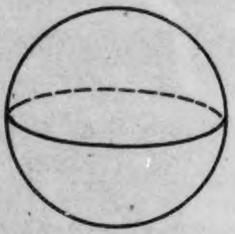

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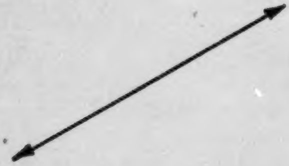

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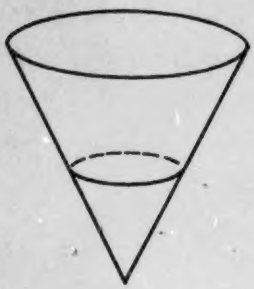

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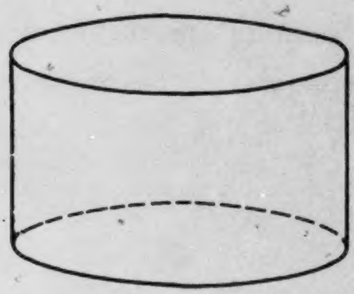


K.

Every _______ figure is also a ________ figure.

The set of all points on a circle is a space figure. So is the set of all points in a circular region. These space figures are also plane figures. The peel covering an orange is a model of a space figure. An ice cream cone is another model. These last two space figures are not plane figures.

1. a. You will remember that simple closed curves are special plane figures. They are those parts which do not cross themselves and which divide the plane in which they lie into three parts: the curve itself, the part called the interior of the curve, and the part called the exterior of the curve.

   b. Which figures shown on the previous page are pictures of simple closed curves? Color the interior of each of these.

2. a. A simple closed surface is a space figure which does not intersect itself and which divides the space in which it lies into three parts: the simple closed surface, the part called the interior of the surface, and the part called the exterior of the surface. The peel covering an orange is a model of a simple closed surface. The part of the orange that you eat is a model of the interior of this surface. All the points of space not covered by the orange make up the exterior.

   Which of the figures on the previous page are pictures of simple closed surfaces? Describe the interior of each of these.

   b. The union of a simple closed surface and its interior is called a space region. The whole orange is a model of a space region.

   Which of the figures are pictures of space regions?

Exercise Set 1

Decide which of the following represent space regions, which represent the interior of a space region, and which represent simple closed surfaces.

1. A walnut, the walnut shell, the walnut meat.

2. A can of peas, the can, the contents of the can.

3. A bottle of soda, the bottle itself with cap.

4. The walls, floor, and ceiling of a room.

5. A block of ice.

6. A jar full of water, the water in the jar, the jar and lid.

7. A hollow rubber ball, the rubber ball and the air inside the ball.

8. An empty shoe box plus the set of points in its interior.

COMPARISON OF SIZES OF OBJECTS

Exploration

Robert bought his father a set of hair brushes for Father's Day. The lady in charge of gift wrapping had to try several boxes before she found one into which the brushes fit properly. How could she decide if the box was too small? too large?

Can your mother fit a 30 pound turkey into the oven in your kitchen?

Do all your text books fit into your brief case?

Is an overnight bag large enough to carry all the clothing you will need for a month's vacation?

Do you think five bus loads of children could fit into one school bus?

In every question raised above, the space occupied by some object was of interest to us. In each case, tell why this was so. We also needed to know when one object was larger than another.

You have already had some practice comparing the sizes of line segments and plane regions. When you have two models of line segments (or plane regions) you have a way of telling whether they are equal. If they are not equal, you can usually decide which one is larger. Can you make this kind of a decision about two space regions? Try the problems in Exercise Set 2.

Exercise Set 2

In each of the following exercises, decide which of the two models represents the larger space region. For each model you should be able to tell which part of the model represents the interior and surface.

1. Your classroom or the school auditorium.
2. A quart jar of milk or a gallon jar of water.
3. An orange or a grapefruit.
4. A family car or a bus.
5. A covered two-quart saucepan or a covered two-quart frying pan.
6. A tennis ball or a golf ball.

Did you know the answer to each of the above exercises at once? Were there some questions for which you were not sure of the answer? How did you decide?

Suppose that we have a pair of shoe boxes. One way in which we could compare two such boxes would be to compare the amounts of contact paper needed to cover the outside of these boxes.

If we make this sort of comparison, what property of the boxes are we interested in?

We might, however, want to compare the amounts that the boxes could hold. Would we need to know the areas of the surfaces of these boxes before we could make this comparison?

Would it help to know the size of the space region enclosed by this box?

Exercise Set 3

In each case tell whether you are interested in the area of the simple closed surface or the size of the space region it encloses.

1. How much paint will you need to cover the inside of a toy chest.

2. Which of two toy chests will provide more storage space.

3. Which of two toys will fit in a gift box.

4. How much material you will need to recover a doll pillow.

5. How many marbles you can put in a box.

6. Make two other examples. Make one of them an example in which you would be interested in the area of a simple closed surface and one in which you would be interested in the size of a space region.

COMPARING SPACE REGIONS

Exploration

Two space regions may be compared by seeing whether one may be included in the other. If one space region can be placed entirely in the interior of the other, the first space region is said to be smaller than the second. This would help us decide that a classroom is smaller than the school auditorium and that a marble is smaller than a beach ball.

We could also decide in this way that an orange is smaller than a grapefruit. Comparing the size of these two pieces of fruit by actually seeing whether one may be included in the other is a little harder. But the orange and the grapefruit are only models of space regions and not the regions themselves. If, without doing too much damage to the peel, we were to remove the parts of the grapefruit that we eat, it is clear that the orange would fit inside the grapefruit.

Exercise Set 4

For each of the following exercises decide which of the two models represents the smaller space region.

1. A baseball or a basketball.
2. A shoe box or a hat box.
3. A milk bottle or a pop bottle.
4. A pear or a banana.
5. A candy bar or an apple.
6. A juice glass full with orange juice or an empty cup.
7. An empty pencil box or a full waste paper basket.
8. A pitcher full with water or a glass full with milk.
9. A glass filled with water or the same glass full with milk.
10. An ice cream cup (dixie cup) or a pint package of ice cream.
11. An empty coffee can or a soup can full with sand.
12. A baseball or a tennis ball.

Were there some exercises for which you were not sure of the answers? Why was it hard to decide? It would seem that it is usually easier to compare space regions which have the same shape.

COMPARING SPACE REGIONS (BY IMMERSION)

Exploration

In some of the exercises we have done it was hard to decide which one of the two given models represented the larger space region. This happened when the two models were of about the same size and shape and our eyes weren't "sharp" enough. Example: An apple and an orange of about the same size; a baseball and a tennis ball; two marbles; etc.

We also had trouble making up our minds when the two models were of such different shapes that it was hard to imagine putting one of them "inside of" the other. Example: A pear and a banana; a candy bar and a marble; etc. We shall now look at another way of using the models of two space regions to decide which model represents the larger region.

These exercises may be answered after having watched your teacher perform this experiment.

1. Take a container partly filled with water. It should be a glass or plastic one so that you can see how high the water is. Make a black mark on the container to show how high the water is.

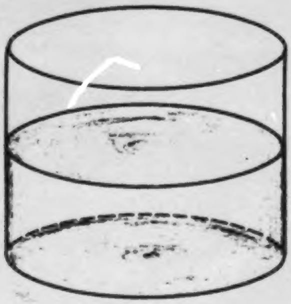


Figure 1

The edges of the container and the water represent a space region. In the picture this region has been shaded. What part of the model represents the simple closed surface? The interior? Could you color the model of the boundary surface blue without coloring any part of the model of the interior? Why not?

Could you color the part of the model representing the interior of the space region red without coloring any part of the model of the boundary? Why not?

2. Take a small rock and a rubber ball. If you drop the rock into the container, will it sink or float? If you drop the ball into the water, will it sink or float?

3. If you were to drop the rock into the water, would you expect the height of the water level to change? If you answered yes, how would you expect the height to change? Why?

Now, gently lower the rock into the water and see what happens.

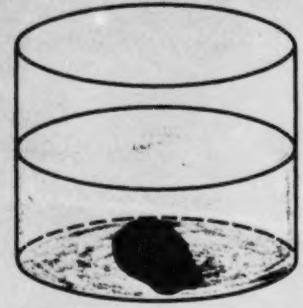


Figure 2

Make a red mark on the container to show how high the water is. Look at the red mark and the black mark. Is the water level as high as it was before? higher? not so high? Was this what you thought would happen? Remove the rock from the water.

4. The rock is a model of a space region. What part of this model represents the boundary of this region? What part of the model represents the interior of this region? Could you color the part of the model which represents the boundary blue?

5. How does the space region represented by the rock compare with the space region represented by the shaded part of figure 1? How do you know?

6. Choose a larger rock than the one you used before. If you were to put this rock into the water, how would you expect the height of the water level to change?

If you have lost any water because of splashing, fill your container up to the black mark. Now gently lower the larger rock into the water. Make a blue mark on the container to show how high the water level is.

Look at the blue mark and the black mark. Is the water as high as it was before? higher? not as high? Can you explain why?

Look at the blue mark and the red mark. When the second rock is in the water, is the water level as high as it was when the first rock was in? Is it higher? Is it lower? Can you explain why?

7. Suppose you were to put both rocks into the water and show the new height of the water level by a green mark. Without actually putting the rocks into the water, imagine what would happen and complete each of the following sentences by filling in the word "above" or "below." For example,

The black mark is below the red mark.

a. The green mark is ______ the black mark.
b. The red mark is ______ the green mark.
c. The blue mark is ______ the green mark.

Now, lower both rocks into the water, make the green mark and check your answers. Remove both rocks from the water.

8. Suppose that you have a third rock and that when you put it into the container, the height of the water level is up to the red mark.

Complete the following sentences by filling in the phrases "larger than," "smaller than," or "just about as large as."

For example: The first rock is a model of a space region which __is smaller than__ the space region represented by the second rock.

a. The third rock is a model of a space region which is ____________________ the space region represented by the first rock.

b. The second rock is a model of a space region which is ____________________ the space region represented by the third rock.

9. If we were to put the third rock and the first rock into the container and indicate the height of the water level by a yellow mark, draw a picture to show where the black, yellow, and red marks would be located.

10. Describe this experiment in your own words. Why do you think this is what would happen?

Try the experiment and check your drawing. Were you right?

Exercise Set 5

In each of the following sentences, cross out the extra words so that the sentence that you have is true. For example,

If rock A is larger than rock B, the space region represented by A is (larger, ~~smaller~~) than the space region represented by B.

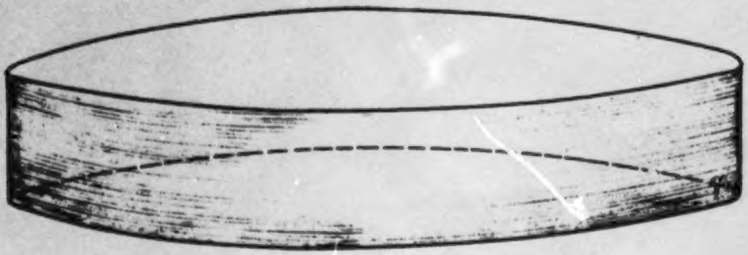
1. If a green rock is the model of a larger space region than that represented by a grey rock, putting the green rock into the container makes the height of the water level (lower, the same, higher) than putting in the grey rock.

2. When we put a red rock into the container, the water level is higher than when we put in a black rock. This helps us decide that the space region represented by the black rock is (smaller than, the same size as, larger than) the space region represented by the red rock. Usually we just say that the black rock is (smaller, the same size as, larger) than the red rock.

3. The higher the water level when we put a rock into the container, the (larger, smaller) the space region the rock represents.

COMMENTS ON OUR EXPERIMENTS

You may have wondered why we wrote "just about as large as" rather than "just as large as" in the exercises you have just completed. Perhaps thinking about the following problem will show you why.



You have probably seen children's backyard pools. Some of these plastic pools are shaped like the one pictured above and are large enough for several children to play in. Suppose that such a pool is half full of water. You could make a mark on the pool to show how high the water is. If you put your first rock into the pool, will the water level rise? Why? You could make a mark to indicate the new height. How should the new mark be related to the old one? Do you think you could tell the difference? Why? Would it help to put the second rock into the pool? (Probably you would not be able to see the difference in the water level. The rise is so small that you could not tell by looking.)

In thinking about the pool, we saw that at times, although there was a change in water level, our eye was not sharp enough to detect it. As a result, if two different rocks raise the water level to the red mark, the best we can say is that they are "about the same size." There could be a slight difference which we could not see.

In all of these experiments, we had to be careful about the kinds of objects we compared. Can you name some kinds of objects it would not be wise to compare by this water method?

MEASURE OF A SPACE REGION

To find the measure of a line segment, we use a unit line segment and see how many such units it takes to cover the given segment.

When we measure a plane region, we use a unit plane region and see how many such units it takes to cover the given region.

Can you guess how we might measure a space region?

What would be a suitable unit?

Should we use a line segment, a unit plane region, a unit space region, or some other new kind of unit?

Exercise Set 6

Tell whether you would use a unit line segment, a unit plane region, or a unit space region to get each of the following measures:

1. The size of the schoolroom floor.
2. The length of a curtain rod.
3. The amount of ice that can fit in a picnic ice chest.
4. The size of the gas tank in the school bus.
5. The size of a mirror.
6. The size of a desk drawer.
7. The size of a packing carton.
8. The height of a door.
9. The size of a tomato juice can.
10. The size of a chalk box.

We see that the measure of a space region is the number of unit space regions it contains.

How could you compare the sizes of the regions bounded by a chalk box and a milk container by using cube blocks?

In order to estimate the measure of the interior of a potato chip can, we can use small cans such as those used in packing tomato paste or tomato sauce. If each child contributes a can, there will be enough to get some measure of the potato chip can.

Why is it wise to select unit cans which have the same size and shape? How could we use these small cans to get a measure of the interior of a potato chip can? of the interior of a shoe box?

Could we use cube blocks to get a measure of the region bounded by the potato chip can? of the space region determined by the shoe box?

In each case decide which gives a better measure--the tomato juice can or the cube block. Why?

VOLUME AND ITS STANDARD UNIT

Exploration

1. Select any two small objects of different shapes (Example: a paper cup, a fruit dish, a cereal bowl, a vase, ...) which can be used to determine models of space regions. Compare the sizes of the two regions by filling the interiors of these objects with marbles. Were all the marbles the same size? If not, explain why this is a disadvantage in comparing the measures.

2. Fill a glass with marbles. See whether the marbles actually fill the interior of the glass. Is the space region determined by a sphere a satisfactory unit for measuring the space region represented by the filled glass? Explain.

3. Could you use marbles to compare the sizes of the space regions represented by a baseball and a grapefruit? Explain your answer.

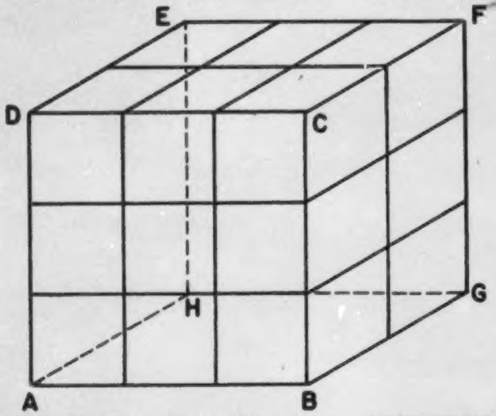
None of these ways seems to be a good way of measuring a space region. Now we are going to see if we can find a better way to get the measure of a space region using a standard unit.

Recall that when we measured line segments and plane regions we might have used any one of a variety of units. However, for purposes of effective communication, a standard unit was selected in each case. We use the inch as a standard unit of linear measure and the square inch as a standard unit of measure for a plane region.

To measure a space region, we will use as a standard unit that space region determined by a cube whose edge is 1 inch long. We call this new unit the cubic inch.

The volume of a space region in terms of this standard unit consists of (1) the measure of this region in terms of this unit and (2) the unit used.

4. Consider the box pictured below. Since 18 cubes, each of edge 1 inch long, fill the box ABCDEFGH, we say the volume of this box is 18 cubic inches.



5. Use cubic inch models to estimate the volume of each of the following:

a. A chalk box.

b. A shoe box.

c. A rectangular baking tin.

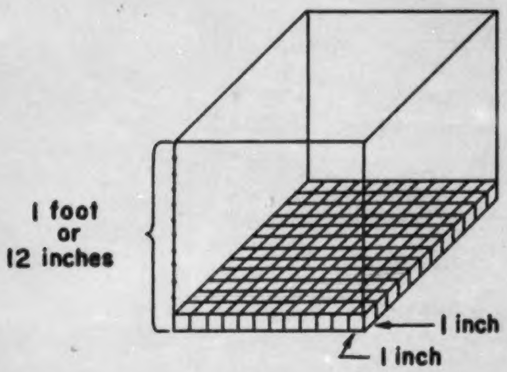
6. It is not always reasonable to use this standard unit, the cubic inch, to find the volume. If we wished to find the volume of a refrigerator-freight car, it would be impractical to use a cubic inch as the unit of measure. For such purposes, we use a cubic foot or a cubic yard. A cubic foot is a space region bounded by a cube, each of whose edges is 1 foot long. A cubic yard is a space region determined by a cube, each of whose edges is 1 yard long.

7. The sketch here should help you see the number of times that a cubic inch is contained in a cubic foot. One layer of cubic inches is sketched.

   a. How many cubic feet are there in this layer?

   b. How many layers will it take to "fill" the big cube?

   c. There are ______ cubic inches in one cubic foot.



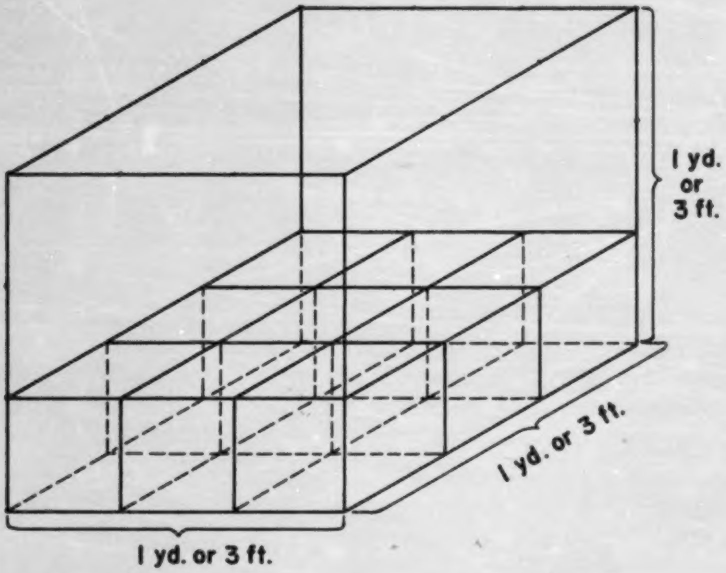
Because there are 12 inches in the height of the cubic foot, we could fill the interior of the cubic foot with 12 layers of the cubic inches, each layer of which is made up of 144 cubic inches. Thus, 144 x 12 = 1728 and we would need 1728 cubic inches to build a model of a cubic foot.

8. Now use the model shown below to help see the number of cubic feet in a cubic yard. Each edge of the big cube is 1 yard, or 3 feet, in length. One layer of cubic feet is sketched. How many layers will it take to fill the big cube? There are ______ cubic feet in one cubic yard.

a. How many cubic feet are there in this layer?

b. How many layers will it take to "fill" the big cube?

c. There are _____ cubic feet in one cubic yard.



9. Can you use your answers to exercises 7 and 8 to determine what the measure of a cubic yard would be in cubic inches?

There are ______________ cubic inches in a cubic yard.

Here is one place where computing with measures is a lot easier than counting up the number of cubic inch blocks needed to build up a model of a cubic yard. Often we can save ourselves a lot of work by using special measures to help compute a volume.

10. In measuring length we sometimes used a standard unit different from the inch, the centimeter. Using the centimeter we can obtain a unit of volume called the cubic centimeter.

    a. What sort of a space region is the cubic centimeter?

    b. Do you remember how many centimeters were needed to cover a length of one meter?

       There are ______ centimeters in one meter.

    c. Describe the space region we would call a cubic meter.

    d. How many cubic centimeter blocks would we need to build a cubic meter? How did you get your answer?

Exercise Set 7

1. Suppose now that we have a cubical chalk box, each edge of which is 5 inches long.

   a. Draw a picture of this box.

   b. What is the volume of this chalk box in cubic inches? How do you know?

   c. Can you describe two ways of solving this problem?

2. Suppose that we have a second chalk box which is one inch higher than the first.

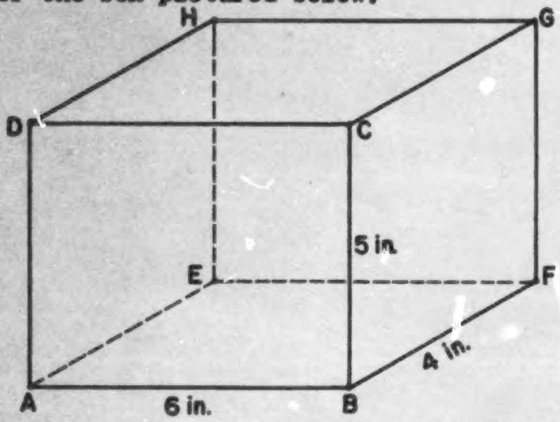
   a. Draw a picture of this box. In your own words, tell which edges of the new box are longer than those of the first box.

   b. What is the volume of this new box in cubic inches?

   c. Describe two ways of finding this volume.

   Which one did you use?

3. Consider the box pictured below.



Its surface is a simple closed surface which together with its interior describes a space region. Edge AB is 6 inches long and is usually called the length of the box. Edge BF is 4 inches long and is called the width of the box. Line segment BC, 5 inches long, is called the height of the box.

a. The chalk box is a model of a figure with ____ faces and _____ edges. Each face is a ______ region bounded by a __________. The surface represented by the chalk box is called a __________________________.

b. Three segments congruent to $\overline{AB}$ are represented by the edges _____, ______, and ______. Each of these is ________ inches long.

c. The edges ______, ______, and _______ represent semgents each of which is congruent to $\overline{BF}$. The measure in inches of each of these edges is ______.

d. The segment BC is congruent to the three segments represented by the edges ______, ______, and _______. All 4 of these segments have a common length of _______.

e. Face ABCD has area _________ square inches.

Another face congruent to ABCD is ___________.

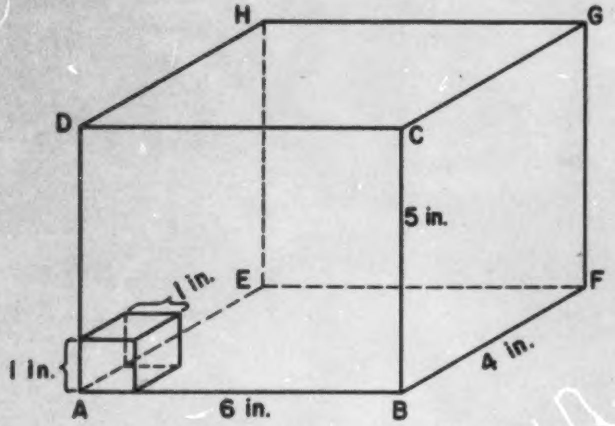
f. Face BFGC has area ________ square inches.

__________ is another face congruent to BFGC.

g. Two other congruent faces are ________ and _________. Each of these has area ____________ square inches.

h. Is the box a cube? _____. How can you tell?

4. Think of putting cubic inch blocks on the floor (face ABFE) of the chalk box of exercise 3. Each block is placed so that it has a face touching the floor.

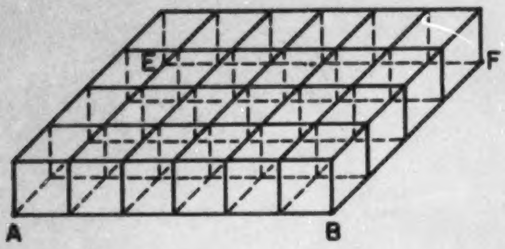


a. The face of the block which touches the floor is a plane region bounded by a ________. Each of its edges has a length of 1 inch.

b. The area of this face is ________ square inch.

c. The area of the floor is ______ square inches.

d. If you fit the blocks as tightly as you can, you would need ______ blocks to cover the whole floor.



e. We would need ________ layers of blocks like the one on ABFE to fill the whole box.

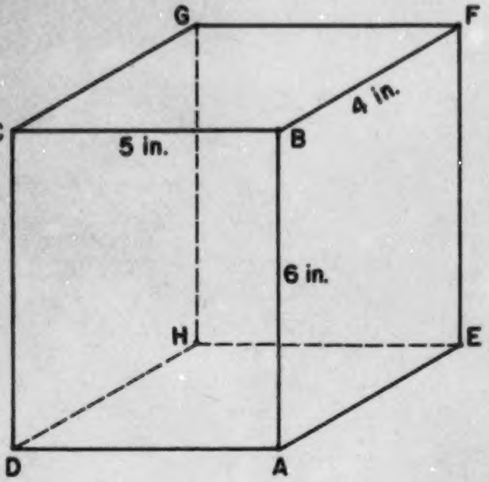
f. Each such layer is built out of ______ blocks.

g. We would need ______ blocks to fill the box.

h. The measure in cubic inches of the space region bounded by the chalk box is ________.

Its volume is __________________.

5. Suppose we now stand the chalk box on another face, say face DAEH.



a. The measure of DA, the length of the box, in inches, is ___________.

b. The measure of the width, AE, in inches is ___________.

c. The height, AB, in inches is ___________.

d. You see that the edges we call the length, width, and height of the box are determined by the face the box is resting on.

Do you think the volume of the region described by the box is the same as it was when the box was resting on face ABFE? Why?

6. a. When the box of exercise 5 is in this position, its floor is face ______.

b. Once again let us imagine that we are covering the floor of the box with blocks which are models of the cubic inch. It will take _____ blocks to cover the floor.

c. Is this the same number of blocks we used before? ____

d. Do you now think the volume of the box is the same as it was when it rested on face ABFE? You can change your mind if you would like to.____________. Why?

e. We will need ___ layers of blocks just like the one used to cover the floor to fill the entire box.

f. There are _____ blocks in each layer. Therefore, we need _____ blocks to fill the whole box.

g. This tells us that the measure in cubic inches of the region bounded by the box is _________.

h. The volume of this region is _______________.

i. How does this volume compare with the volume we got when the box was resting on face ABFE? ________.

7. a. Do you think that if we let the box rest on face ABCD the volume would be changed? ______. Why? ________.

b. Tell how you could use layers of cubic inch blocks to show that your answer is correct.

8. a. Do you see a quick way to find the volume of the chalk box using the measures in inches of the length, width, and height?

b. Does your way depend upon which face is the floor of the box?

c. Can you explain why your way works?

d. Say in words what measures you would like to know to find the volume of a space region bounded by a rectangular prism.

9. a. Is a cube a rectangular prism?

b. Is every rectangular prism a cube? Why?

c. How could you find the volume of a cube?

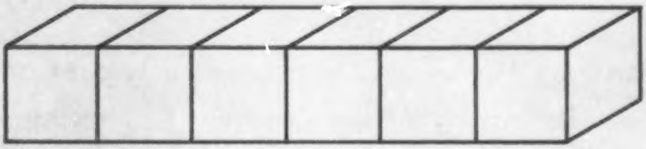
d. Tell in words which measures you would like to know to find the volume of a cubical space region.

e. Is this an easier or a harder problem than finding the volume determined by a box which is not a cube? Why?

Exploration

The volume of the space region bounded by a prism is often just called the volume of the prism. In the same way, we often speak of the volume of a cube when we really mean the volume of the space region bounded by the cube. We can use the shorter expression whenever we are fairly sure that we will not be misunderstood.

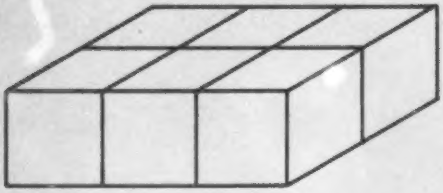
1. If we place 6 cubes as in the diagram below, we have built a model of a rectangular prism.



   a. What is its volume? Observe that this model is 6 units long, 1 unit wide, and 1 unit high.

   b. Are the units we refer to linear units, area units, or volume units?

2. Another model of a rectangular prism which could be made from the same six cubes would look like this:



a. What is the volume of the above model?

b. What are the length, width, and height?

c. Can you build still another model of a rectangular prism using these six cubes?

d. What are the length, width, and height of your new rectangular prism?

e. The length, width, and height are expressed in what kind of units?

f. The volume, however, is expressed in

____________ units.

3. Choose 24 unit cubes and build at least 6 different models of space regions bounded by rectangular prism. Keep a record of the following measures. Remember that the length, width, and height are measured in one kind of unit and the volume in another.

	Measure of Length	Measure of Width	Measure of height	Measure of Volume
a.				
b.				
c.				
d.				
e.				
f.				

g. Do you see any relationship between the measures of the length, width, and height of the prism and the measure of its volume?

h. Was this what you expected?

i. Can you explain what you observed?

Exercise Set 8

1. List 5 sets of possible lengths, widths, and heights of rectangular prisms whose volume is 36 cubic inches. Do not actually build models of these prisms.

2. List 5 sets of possible lengths, widths, and heights of rectangular prisms whose volume is 144 cubic feet.

3. Think of a cube whose edge is 1 centimeter long. What do you think would be a suitable name for this unit of measure of a space region?

   List 5 sets of possible lengths, widths, and heights of rectangular prisms whose volume is 100 cubic centimeters.

4. If the length, width, and height are in inches, the volume is in ________ ________. If the length, width, and height are given in feet, the volume is in _______ _______. If the length, width, and height are given in centimeters, the volume is in ________ ________.

5. Complete the following table. All measures refer to space regions bounded by rectangular prisms.

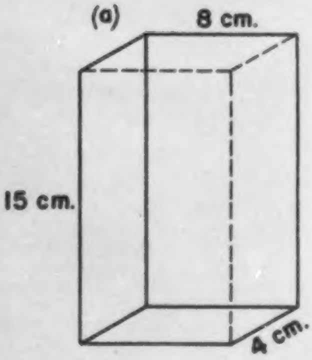
	Measure of Length of Prism	Measure of Width of Prism	Measure of Height of Prism	Measure of Volume of Space Region
a.	5	8	3	___
b.	2	6	8	___
c.	12	4	1	___
d.	___	6	7	168

6. To calculate the measure of a solid region bounded by a rectangular prism, multiply the measures of the length, width, and height of the rectangular prism. If a, b, c, are the number of units in the length, width, and height of a rectangular prism, the volume of the prism is __________.

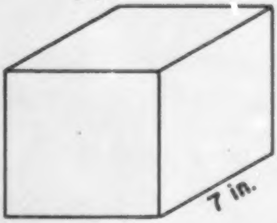
7. Show how you could use an exponent to write the product for calculating the volume of a cube of edge 5 inches, of edge 2 feet, of edge a centimeters.

8. Calculate the volume in each case, if enough information is given.

(a)

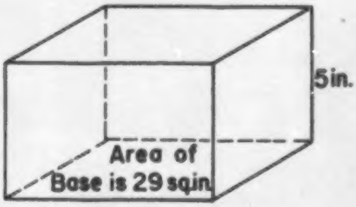


(b)



This figure is a Cube

(c)

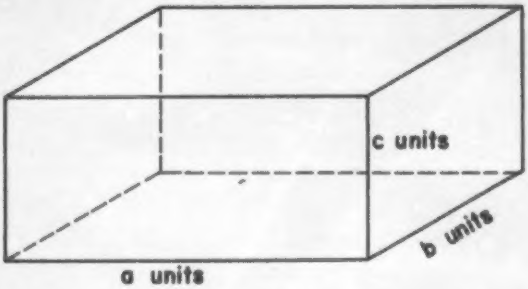


COMPUTING THE VOLUME OF A SPACE REGION BOUNDED BY A RIGHT PRISM (BOX)

Exploration

Were you able to calculate the measure of the space region in exercise 8 (c) of the last set of exercises? Here is one way you might have thought about this problem.

If the measure of the length, width, and height of a rectangular prism are a, b, and c, then the measure of the volume is $a \times b \times c$. The measure of the length, width, and height is, of course, the number of linear units needed to cover each of these line segments. The measure of the volume, however, tells us how many cubic units there are in the space region bounded by the prism. All four of these measures are numbers. If we let V stand for the measure of the volume, we can write what we have discovered as $V = (a \times b) \times c$.



The product $(a \times b)$ gives the measure of the rectangular region which is the base of the prism. Another way of saying this is that $a \times b$ tells us how many units we need to cover the base of the prism. Therefore, in problem 8 (c) we may write:

$$V = (\text{measure of the base}) \times c$$
$$V = 29 \times 5$$
$$V = 145$$

The volume is 145 cubic units.

Exercise Set 9

1. Calculate the volume of a space region bounded by a rectangular prism, using the following information:

	Area of Base	Height	Volume
a.	42 sq. cm.	15 cm.	______
b.	38 sq. in.	10 in.	______
c.	100 sq. in.	4 ft.	______

2. If the volume of a solid region bounded by a rectangular prism is 144 cubic units and the height 10 linear units, what is the area of the base?

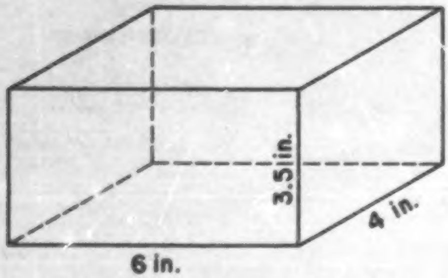
3. How many packages of paper napkins can be packed in a carton whose base is a 2 ft. square, if the carton is 3 ft. high? Each package of paper napkins is 6 in. by 6 in. by 4 in.

4. Space is needed in a classroom to hold a set of textbooks. Each textbook is one inch thick and has a cover 6 in. by 8 in. Can a space 1680 cu. in. hold 35 texts?

Exploration

Let us look at a box where the length of one edge is not a whole number of units.

1. Look at the box pictured below.



Its length is ________

Its width is ________

Its height is ________

a. If you use blocks which are models of the cubic inch, how many such blocks would you need to cover the floor?

b. The area of the floor of the box is ________.

c. If you used two such layers of blocks, would you exactly fill the box?

d. Would two layers contain too many blocks, just enough blocks, or too few blocks to fill the box?

e. This tells us that the volume of the box is ______ 48 cubic inches.

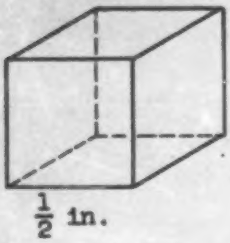
f. Would 3 such layers exactly fill the box?

g. Would 4 such layers exactly fill the box?

h. We see that the volume of the box is ______ 72 cubic inches and ______ 96 cubic inches. This is the best we can do using cubic inch blocks.

i. What sort of blocks could we use to get a better estimate of the volume of this box?

2. Suppose we have a cubical block of edge $\frac{1}{2}$ inch.



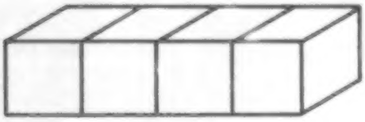
a. The volume of the space region for which this block is a model is ______________.

b. It would take ______ of these blocks to build a model of the cubic inch.

3. a. Let us use these smaller blocks to cover the floor of the box of exercise 1. We will pack these blocks in tightly just as we did before and in such a way that each block has a face on the floor. The face of the block is a model of a __________ bounded by a _____ of area __________.

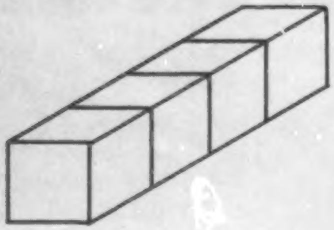
b. We would have to pack in _____ such blocks to cover one square inch of the floor.

4. a. If we line the blocks up as shown below,



it would take _______ of these $\frac{1}{2}$ inch blocks to build a model as long as the box.

b. If we build columns of blocks like this,



it would take _____ such blocks to build a prism as wide as the box.

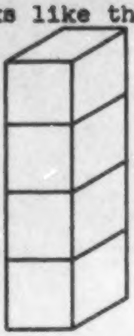
c. Using these facts we see that it would take _____ of our new blocks to cover the entire floor of the box.

d. Every 4 such blocks would cover ______ sq. in. of the floor.

e. This shows that our 96 blocks will cover ______ sq. in.

f. The area of the floor of the box is ____________________.

5. a. If we pile up blocks like this,



it will take ______ such blocks to build a prism as high as the box.

b. This helps us conclude that we will need _____ layers of blocks just like the one on the floor to fill the whole box.

c. Each such layer contains ______ blocks and therefore we will need ___________ or _______ blocks to fill the box.

d. The measure of our box in terms of this new unit is ________.

e. If we now remember that it took 8 of these blocks to build a model of ______ cu. in., we see that the volume of the box is _____________ cu. in. or ________ cu. in.

6. We also see that this volume of the box of exercise 5 is less than 96 cubic inches and bigger than 72 cubic inches. This was pointed up by the discovery that 3 layers of 24 cubic inch blocks each were not quite enough to fill our box, while if we built 4 such layers we had too many blocks.

a. Using these smaller blocks we can get ______________ estimate of the volume.

b. In general, using a smaller unit will lead to a more ________________________________ estimate.

c. For the box in question, the measure (in inches) of the length, a, is ________________.

d. The measure (in inches) of the width, b, is _______.

e. The measure (in inches) of the height, c, is _______.

f. $a \times b \times c$ is __________. Is this the measure, V, (in cubic inches) of the volume?

g. Will our formula $V = a \times b \times c$ work for this box?

Exercise Set 10

1. Consider a box of length $6\frac{3}{4}$ inches, width 4 inches, and height 3 inches.

   a. Draw a picture of this box.

   b. What are a, b, and c for the prism represented by this box?

      a = _____ b = _____ c = _____

   c. Our formula leads us to believe that V = ______ and that the volume of the space region bounded by this prism is ____________.

2. Could you check your answer by building up layers of cubical blocks of edge 1 inch? Explain.

3. Could you check your answer by building up layers of cubical blocks of edge $\frac{1}{2}$ inch? Why?

If we studied more rectangular prisms, we would find that we could always use the formula, $V = a \times b \times c$ to help us find the measure of the volume.

Another way of thinking of this formula is

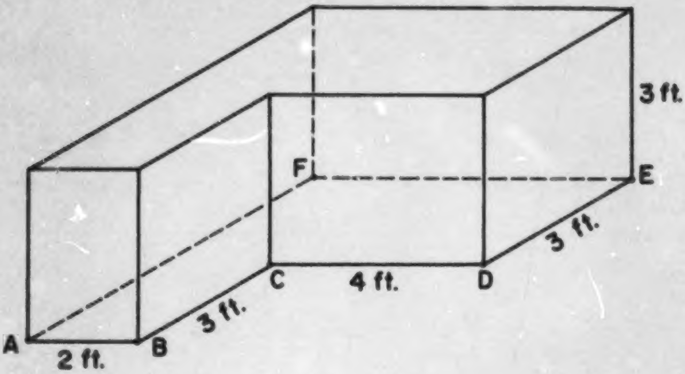
$$V = (a \times b) \times c \quad \text{or}$$

Measure of volume (in cubic inches) = Measure of base (in square units) × height (in linear units)

VOLUME OF SOME SPACE REGIONS WHOSE BASES ARE NOT RECTANGLES

Exploration

1. a. Consider the space figure pictured below. (A model of such a figure might be a storage chest specially designed to fit into a corner of a room.) Can we find the volume of the region bounded by this figure?

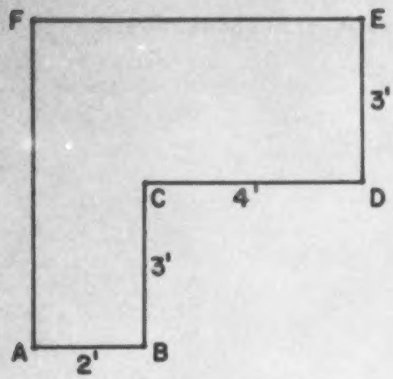


b. Since the measures of the edges are expressed in terms of feet, what unit would we expect to use to express the volume?

c. One model of this unit would be a cubical block. What would be the length of the edge of such a cube?

d. If we pack these cubes as usual, how many such cubes would it take to cover the floor of such a chest?

e. If you have trouble answering this question, you might imagine making a trace of the outline of the floor of the chest. Your trace would look like the figure at the top of the next page.



2. Look at the outline shown above of the floor of the chest.

   a. How long is $\overline{AF}$?

   b. How long is $\overline{FE}$?

   c. Does this help you answer your question?

   d. The area of the floor of the chest is ________________.

   e. How many such layers of blocks would you need to fill the box?

   f. The measure of the volume of the chest is _________ (in cubic feet) or ____________ (in cubic feet).

V, the measure of the volume, is the measure of the base (in square feet) multiplied by the measure of the height (in feet).

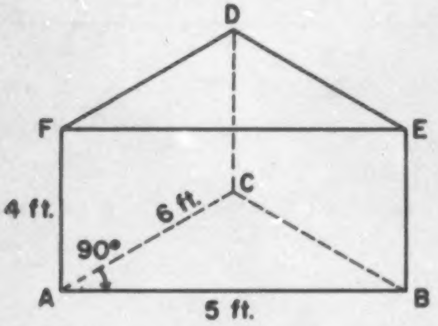
$$V = B \times h$$

This was the same formula that we used for the chalk box. The base of the chalk box was a plane region bounded by a rectangle whose sides had measures a and b where a and b represent numbers. The measure of the base (in square units), therefore, was $a \times b$. If the height of the box is c, $V = B \times h$ becomes $V = (a \times b) \times c$. The letters a, b, c represent numbers.

3. A storage container is pictured below. It is a model of a space region.

a. The base of the space region is a __________ bounded by a ______________________.

b. If again we use blocks which are models of the cubic foot, we will need ______ layers of such blocks to fill the container.



c. Can we exactly cover the floor using these blocks?

d. What is the area of the plane region which has the floor of the chest as its model?

e. Could you cover this floor of the chest using your cubic-foot blocks?

f. Can you imagine such a covering?

g. How many blocks would you need?

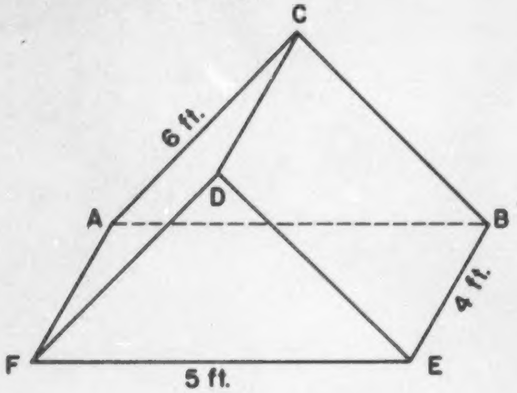
h. To fill the entire container we would need how many blocks?

i. Does our formula, V = B × h, work for this space figure?

j. What is the volume of the region described by the picture?

4. a. Face ABEF of the container is a model of a plane region bounded by a ____________________.

b. Suppose we let the model rest on this face. Do you think the region now bounded by this figure will have a different volume? Why?



5. a. If the container is in this position, its floor is the model of a plane region of area ____________________.

b. Could we use

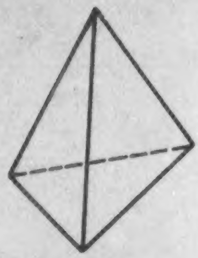
$$V = Bh = 20h$$

where h is the measure (in feet) of the height from the base up to edge DC to find the measure of the volume? Why?

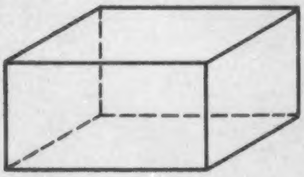
Exercise Set 11

1. Which one of the following pictures represent space regions the measure of whose volumes could be found by $V = Bh$? Why?

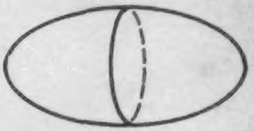
a.



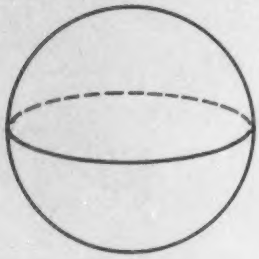
b.



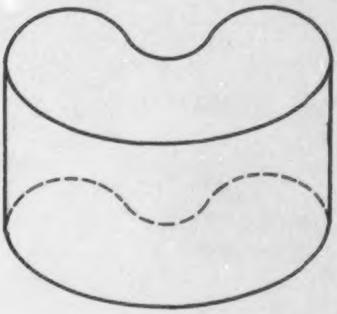
c.



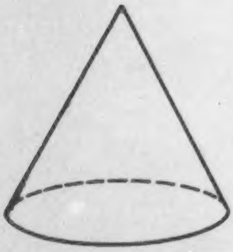
d.



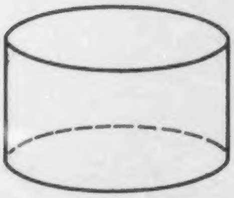
e.



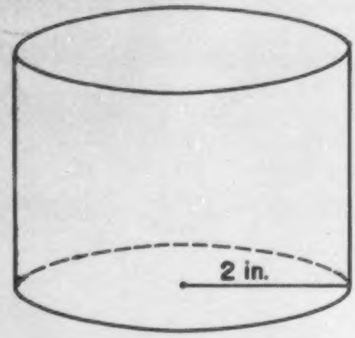
f.



g.



2. Consider the space region pictured below.



a. A model of such a space region might be a tin can. Do you think you could find the measure of the volume of this region by means of the formula, $V = Bh$?

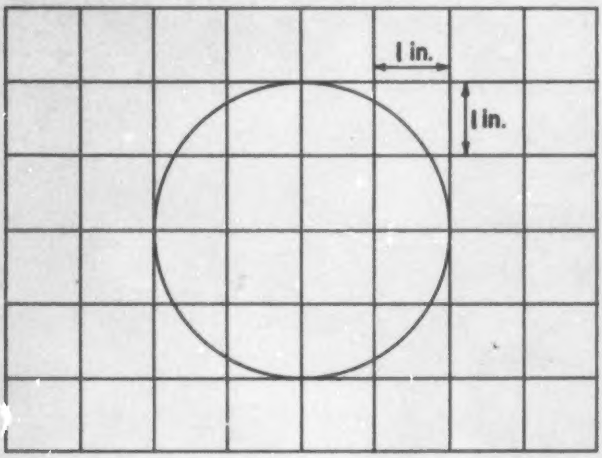
b. How did you make your decision?

c. What unit would you expect to express this volume? Why?

d. What is the name of the surface bounding this space region?

e. What is measure of h, in inches, for this region?

f. One way of estimating B is to use a rectangular grid as we did in the unit on area. We could use our model of the region and make a trace of the boundary of its base. This boundary curve is a circle of radius 2 inches. Your trace would look like this:



A grid with 1 inch squares has been put on your trace. What is the measure (in square inches) of the plane region bounded by a circle of radius two inches? (Use your grid to estimate this measure.

g. Using this grid, our smallest estimated value of B is ______ and the value of V that goes with it is ______.

h. Our largest estimated value of B is ________ and the value of V that goes with it is ______________.

i. The volume of the space region is smaller than _______ and greater than _________.

j. How could we get a better estimate of the volume?

We have seen that some space regions -- those whose models can be built up out of equal layers -- are such that the measure of their volume is given by

$$V = B \times h$$

or

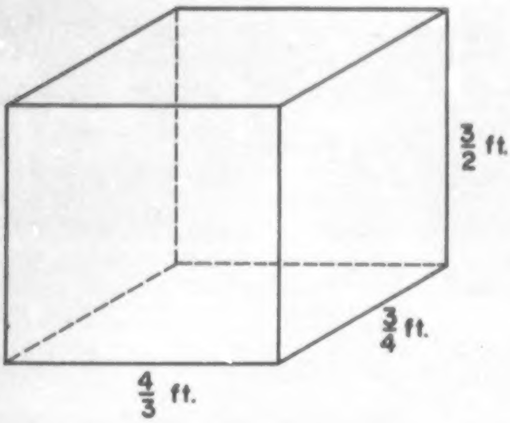
Measure of Volume (in cubic units) =

Meausre of Base (in square units) ×
Measure of height (in linear units)

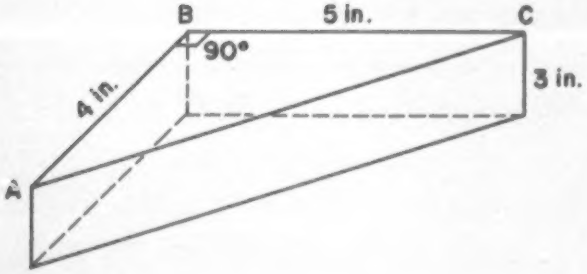
There are other space regions the measure of whose volume cannot be found this way.

Exercise Set 12

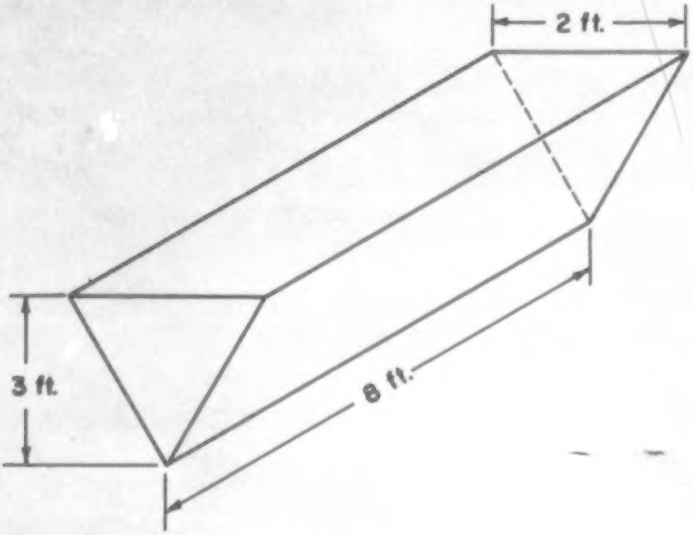
1. A hat box is a model of a space region. What is the volume of this region?


$\frac{3}{2}$ ft.
$\frac{3}{4}$ ft.
$\frac{4}{3}$ ft.

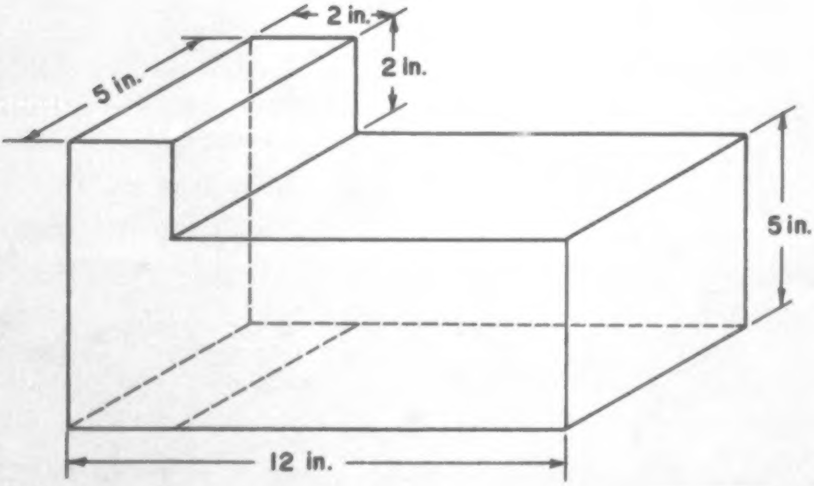
2. A plastic dish which can hold a piece of pie is pictured below. What is the volume of the space region represented by this dish?


B
5 in.
C
90°
3 in.
4 in.
A

3. What is the volume of the water trough pictured below?



4. Find the volume of the space region bounded by this space figure.


2 in.
2 in.
5 in.
5 in.
12 in.

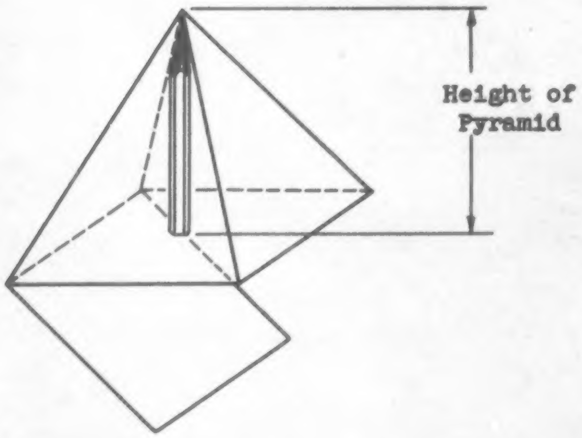
COMPARISON OF VOLUMES OF SPACE REGIONS BOUNDED BY RECTANGULAR PRISMS AND PYRAMIDS

Exploration

1. Now look at the pyramid which your teacher has. Find the linear measures (in inches) of the sides of the base of this model. Record your measures under the first two headings in the table below.

Measure (in inches) of the length of the base	Measure (in inches) of the width of the base	Measure (in inches) of the height of the pyramid	Measure (in cubic inches) of the volume

2. What is the height of the pyramid? Your teacher will now show you how to estimate it.



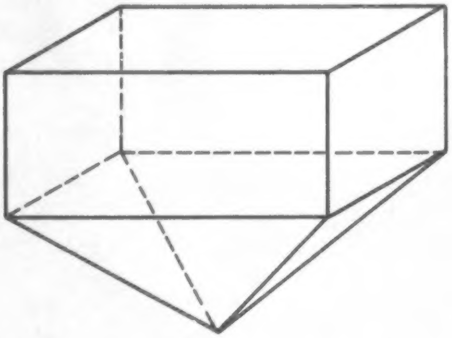
3. a. How do the measures of the cube compare with those of the pyramid?

   b. Which space region do you think is larger, that bounded by the prism or that bounded by the pyramid?

   c. Your teacher will fill the pyramid with rice. How many pyramids of rice does it take to fill the interior of the cube?

   d. Can you estimate the volume of the pyramid? Record this estimate under the fourth heading in your table.

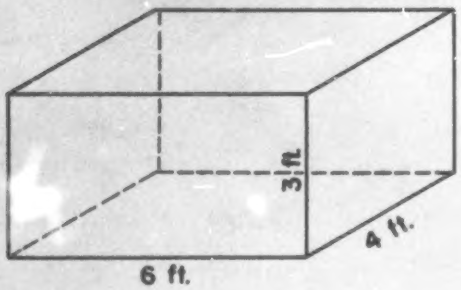
4. If a rectangular prism and a rectangular pyramid have congruent bases and heights of equal measure, how do you think the volumes bounded by these surfaces will compare?

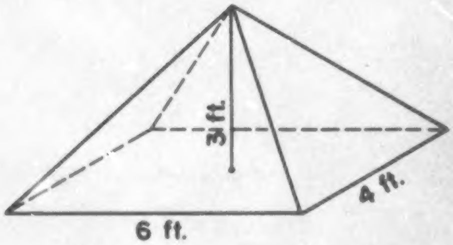


5. a. We have performed an experiment which showed that if a prism and a pyramid have heights of the same measure and if their bases are bounded by congruent rectangles, the volume of the _____ is three times the volume of the ____________.

   b. Another way of saying this is that the volume of the __________ is one-third the volume of the __________.

6. Consider the prism and the related pyramid pictured below.





a. Since the length, width, and height are expressed in feet, we will expect to express the volume in ________________.

b. The heights of the two figures are ____________.

c. The bases of the two figures are ___________.

d. The volume of the pyramid is ______________ the volume of the prism.

e. Another way of saying this is that the volume of the prism is ______________ the volume of the pyramid.

f. The volume of the prism is ___________________.

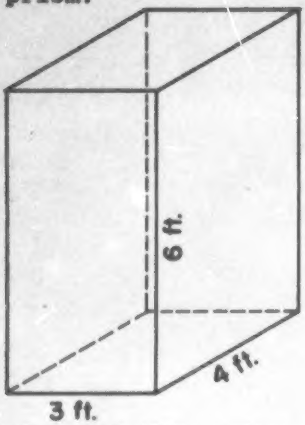
g. From this we can conclude that the volume of the pyramid is ________________.

7. a. We know that if a, b, and c are the measures of the length, width, and height of a prism, the measure, V, of the volume is given by the formula V = ______________.

b. If we have a pyramid with length of measure a, width of measure b, and height of measure c, the measure of its volume will be given by the formula V = ______________.

c. Tell what this formula says in words.

8. Now consider this prism.



a. How does the volume of this prism compare with the one we had before?

b. Does this seem reasonable?

c. How does the volume of our pyramid compare with the volume of this prism?

d. What are a, b, and c for the new prism?

e. For the prism of exercise 6?

f. For the pyramid of exercise 6?

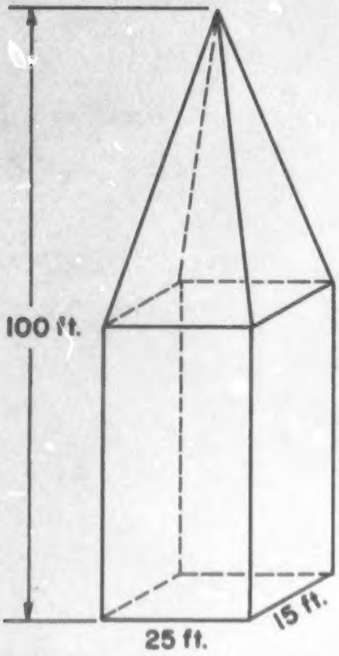
g. Does $V = abc$ give you the measure of the volume of each prism.

h. Does $V = \frac{1}{3} a \times b \times c$ give you the measure of the volume of the pyramid?

10. What is the volume of a pyramid of length 10 inches, width 4 inches, and height 1 foot? Express your answer in terms of cubic inches.

Exercise Set 13

1.



What is the measure
in cubic feet
of this monument?

2. The volume of a rectangular prism is 216 cubic inches. The height is 9 inches.

   a. What is the area of the base?

   b. Name some pairs of numbers which could be its length and width, in inches.

3. The volume of a rectangular pyramid is 216 cubic inches. The area of the base is the same as that of the prism of exercise 2. What is the height of this pyramid?

Chapter 8

ORGANIZING AND DESCRIBING DATA

INTRODUCTION

It would be a very tedious job to study the score card of every baseball game in which Mickey Mantle has played in order to determine how many times he has been at bat and how many times he has obtained hits. But we know at a glance how successful he is as a batter when we say his batting average this year is .326, _if_ we know what "average" means, and whether .326 is "good."

There are many, many facts about today's world. Many of these facts are expressed in terms of numbers. There are facts about people's heights and weights, facts about baseball, facts about heights of buildings, facts radioed from satellites in outer space, facts about population, and many others.

In this chapter you are going to study some ideas about organizing and describing data. You are going to learn that the word "average" has several meanings and that sometimes an "average" is very misleading. You are also going to learn various ways of presenting facts with graphs.

ORGANIZING DATA

Exploration

The teacher and the boys and girls in a sixth grade class wanted to buy a pet for their room. After some discussion, they decided to have each person write his choice on a slip of paper. They could choose from one of these: guinea pig, turtle, lizard, hamster, or garter snake.

When the slips were collected, they looked like this:

HAMSTER	TURTLE	GARTER SNAKE	LIZARD	GARTER SNAKE	GUINEA PIG	HAMSTER	GARTER SNAKE	GUINEA PIG
HAMSTER	GARTER SNAKE	HAMSTER	GUINEA PIG	GARTER SNAKE	GARTER SNAKE	HAMSTER	GUINEA PIG	HAMSTER
GARTER SNAKE	LIZARD	LIZARD	HAMSTER	HAMSTER	HAMSTER	TURTLE	GARTER SNAKE	GARTER SNAKE

1. How could the children decide which animal to buy? Could they tell by just glancing at the votes or did they need to organize the information given by these votes? Would the class need to vote again?

When we think of arranging information in some way so that it can be more easily understood, we say we are organizing data. Data is another name for information or groups of facts.

2. A committee counted the votes for the classroom pet. Then they made a table showing the facts given by the votes.

It looked like this:

TABLE 1

Votes for Class Pet	
Guinea Pig	4
Turtle	2
Lizard	3
Hamster	9
Garter Snake	9

A table always has a title to show what the facts are about.

a) Is it easier to tell by looking at this table or by looking at the slips of paper, how many votes each animal received?

The sixth graders decided to vote again to choose between a hamster and a garter snake. After the votes were counted, a table was made:

TABLE 2

Final Vote for Class Pet	
Hamster	15
Garter Snake	12

b) Tell how a table helps you understand a set of facts.

3. George, Harry, Jim, and John each have a newspaper stand. The number of papers they sold one week was: Monday - George 3, Harry 28, Jim 15, John 36; Tuesday - George 10, Harry 29, Jim 21, John 38; Wednesday - George 18, Harry 29, Jim 47, John 38; Thursday - George 30, Harry 30, Jim 47, John 40; Friday - George 52, Harry 32, Jim 47, John 42; Saturday - George 70, Harry 60, Jim 50, John 44; Sunday - George 98, Harry 75, Jim 54, John 45.

Without organizing the data, answer these questions:

a) On what day did Jim sell as many papers as George and Harry together?

b) Which boy sold the same number of papers three days in a row?

c) Which boy sold more papers each day than he sold the day before?

d) Which boys sold more papers, or at least as many as the day before, each day through the week?

e) Which boy sold more than three times as many papers one day as he did the day before?

f) Which boy's sales for each day of the entire week did not vary over 10 papers?

Now look at the data organized in this table. Does the table make it easier to answer the questions?

TABLE 3

Newspaper Sales for One Week

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
George	3	10	18	30	52	70	98
Harry	28	29	29	30	32	60	75
Jim	15	21	47	47	47	50	54
John	36	38	38	40	42	44	45

4. Eight Girl Scout troops in Centerville sold boxes of cookies. There were three kinds of cookies, one kind to a box. There were boxes of mint, chocolate, and vanilla cookies. The sales, in boxes, of each troop were: Troop 1 - mint 48, chocolate 63, vanilla 35; Troop 2 - mint 34, chocolate 27, vanilla 30; Troop 3 - mint 72, chocolate 51, vanilla 40; Troop 4 - mint 25, chocolate 14, vanilla 12; Troop 5 - mint 75, chocolate 39, vanilla 51; Troop 6 - mint 51, chocolate 62, vanilla 37; Troop 7 - mint 132, chocolate 98, vanilla 99; Troop 8 - mint 82, chocolate 98, vanilla 76. Make a table to show the cookie sales. Use the table to help you answer these questions:

a) Which troop sold the most cookies? (Would it help if you showed the total sales in the table?)

b) How many more boxes of chocolate cookies were sold than vanilla? (Would it help to show the totals of each kind of cookie sold?)

Exercises a) and b) show that it sometimes helps to total some of the data given in a table and to show this total in the table.

c) Which troop sold the most boxes of chocolate cookies? the least?

d) Look at the part of the table which shows the number of boxes of mint cookies sold by each troop. Troop 7 sold one less box of mint cookies than the sales of two other troops combined. Name these two troops.

e) Make up some questions of your own that can be answered by studying the table.

SUMMARY

We have learned:

1. Data is another name for information or groups of facts.
2. We organize data to make it more easily understood.
3. We can organize data by using tables.
4. Tables have titles to help us understand the data.
5. It is sometimes helpful to add the numbers given in tables and show the sums in the table.

Exercise Set 1

1. Linda and Perry's arithmetic scores for two weeks are given below. The scores tell the number of correct answers.

Linda: Monday - 18, Tuesday - 54, Wednesday - 12,
Thursday - 44, Friday - 96; Monday - 7, Tuesday - 75,
Wednesday - 20, Thursday - 35, Friday - 72.

Perry: Monday - 20, Tuesday - 54, Wednesday - 12,
Thursday - 43, Friday - 97; Monday - 6, Tuesday - 77,
Wednesday - 19, Thursday - 36, Friday - 70.

Make a table to show these facts.

Which of these sixth-graders had the higher total score.

2. Six boys are going on a hike. They are going to take the following things with them:

Bill - canteen, ax, cookies
Chuck - sandwiches, canteen, cookies
Lee - knife, compass, wieners
Dick - beans, canteen, first aid kit
Jack - matches, sandwiches, wieners
Sam - buns, beans, canteen

Make a table to show the number of boys who are taking canteens, cookies, etc.

a) How many different items are the boys taking?

b) How many boys are taking wieners? buns? beans? canteens?

3. The "home states" of the Presidents of the United States are: Washington - Virginia; J. Adams, Massachusetts; Jefferson - Virginia; Madison - Virginia; Monroe - Virginia; J. Q. Adams - Massachusetts; Jackson - South Carolina; Van Buren - New York; W. H. Harrison - Virginia; Tyler - Virginia; Polk - North Carolina; Taylor - Virginia; Fillmore - New York; Pierce - New Hampshire; Buchanan - Pennsylvania: Lincoln - Kentucky; Johnson - North Carolina; Grant - Ohio; Hayes - Ohio; Garfield - Ohio; Arthur - Vermont; Cleveland - New Jersey; Harrison - Ohio; McKinley - Ohio; T. Roosevelt - New York; Taft - Ohio; Wilson - Virginia; Harding - Ohio; Coolidge - Vermont; Hoover - Iowa; F. D. Roosevelt - New York; Truman - Missouri; Eisenhower - Texas; Kennedy - Massachusetts.

Make a table to show the number of presidents from each state.

a) How many presidents were from the six New England states?

b) How many presidents did Ohio and Virginia together furnish?

c) How many presidents were from west of the Mississippi River?

d) What 3 states furnished over half of the presidents?

e) Name the states in which more than two presidents were born.

ROUNDING NUMBERS:

There are times when we need to know the exact number of objects in a set. At other times we need only to know "about how many." The swimming instructor finds it necessary to know the exact number of children that go into the pool. The same number of children should come out, at the end of the period, as went in it at the beginning of the period. It is not enough for him to know only "about how many" children there are.

There are other numbers that need to be known exactly. Some numbers such as your telephone number, your locker number, and your house number need to be stated exactly.

There are many times when we can use a number which tells "about how many." Most measurements can be made only with a certain degree of accuracy. These "about how many" numbers are sometimes called estimates, approximations, or rounded numbers.

We use rounded numbers at times when we need only an estimate. The police estimate a crowd of people at a parade at 200,000. The traffic officer may estimate the speed of the passing auto at 60 miles per hour. These estimates are given in round numbers. What other examples can you give of numbers which tell "about how many?"

Sometimes we replace a count or measurement of something by a less accurate one. It might not be important to use the number of people in the United States as 179,323,175. This number was given by the census of 1960. Do you think this number is exactly correct? Would we be correct if we said there were nearly 180,000,000 people in the United States in 1960?

Suppose we wanted to know what part of the people of the United States lived in the state of New York in 1960. The census reported that 14,759,429 people lived in New York. Do you think this number is exactly correct? Would 15,000,000 be nearly correct? If we used one of these fractions to tell us what part of the U. S. people lived in New York, which one would be easier to write in a simpler form:

$$\frac{15,000,000}{180,000,000} \quad \text{or} \quad \frac{14,759,429}{179,323,175}$$

There are many other times when we should use rounded-off numbers in place of exact measures. Think of this example. Suppose you are making a trip in your car. On the first part of the trip you read from the odometer that you travel 121.7 miles. On the second part of the trip you forget to read the odometer. But the map gives the mileage between 95 and 105. Now what numbers can you add to find how far you traveled on the trip? Should you use 121.7 miles for the first part of the trip? What mileage would you use for the last part of the trip? If you round off 121.7 to the nearest multiple of ten the number would be 120. The multiple of ten between 95 and 105 is 100. If you said that you traveled 120 + 100 or 220 miles on the entire trip, would 220 be about right?

There are several ways of "rounding off" numbers. For our work here, we will use a very simple way, as shown on the following page.

Below is a number line:

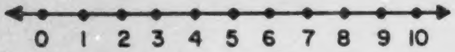


Figure 1

Let us use only the segment from 0 to 10. What is the middle point of this segment of the number line?

We sometimes round off numbers to the nearest multiple of 10. If we round off the number 7 to the nearest multiple of 10, we say it is rounded to 1 ten. The number 4 is rounded to 0 tens.

Round off 6 to the nearest multiple of 10.

What happens if we round off 2 and 3 to the nearest multiple of 10?

In rounding off to the nearest multiple of ten, what do you suppose happens to any number which is to the left of point 5 on the number line?

What would a number to the right of five become?

How do we round off the number five to the nearest multiple of ten? (We shall round off 5 to 1 ten as a rounded number.)

Below is another section of a number line:

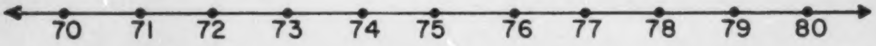


Figure 2

How would we round off the numbers 70, 71, 72, 73, 74 to the nearest multiple of ten?

How would we round off the numbers 75, 76, 77, 78, 79 to the nearest multiple of ten?

Round these numbers to the nearest multiple of ten.

a) 147

b) 132

c) 984

d) 565

e) 2,168

f) 995

In rounding off to the nearest multiple of ten, what do we do when the number 5 is in the one's place? (We replace the number by the next multiple of 10 which is greater than the given number; e.g., 5 would be rounded to 10, 15 rounded to 20; 135 rounded to 140, etc.)

If we round off a number to the nearest multiple of ten, we replace the number by the nearest multiple of 10.

We round off 16 to 20, 37 to 40, 89 to 90, 81 to 80, 176 to 180.

If the number to be rounded off to the nearest multiple of 10 is halfway between two multiples of 10, we replace the number by the next larger multiple of 10.

We round off 15 to 20, 25 to 30, 115 to 120, 905 to 910, etc.

If we round off a number to the nearest multiple of one hundred, we replace it by the nearest multiple of 100.

We round off 46 to 0, 89 to 100, 176 to 200, 341 to 300, etc., if we are rounding off to the nearest multiple of 100.

If the number to be rounded off to the nearest multiple of 100 is halfway between two multiples of 100 we replace the number by the next larger multiple of 100.

We round off 50 to 100, 150 to 200, 250 to 300, etc., if they are to be rounded off to the nearest multiple of 100.

Now do you see how to round off a number to the nearest multiple of 1000? of 10,000, etc.?

In the section of a number line below we have the numbers by tens from 100 to 200.

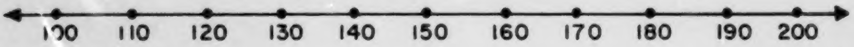


Figure 3

What is the middle number between the numbers 100 and 200?

Round off 140 to the nearest multiple of one hundred.

Round off 180 to the nearest multiple of one hundred.

If the number in the tens' place is less than five as in 110, 120, 130, 140, what is the nearest hundred?

If the tens' place digit is 5 or greater, what is the nearest multiple of one hundred on the above number line?

Do you suppose this holds true for rounding off all numbers to the nearest multiple of hundred?

Round off these numbers to the nearest multiple of one hundred.

a) 92
b) 550
c) 602
d) 3,849
e) 1,841,731

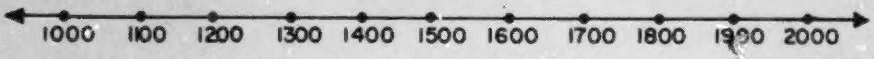


Figure 4

Dick had the above diagram to look at. He was asked to round off the following numbers to the nearest multiples of one thousand: 1400, 1700, 1500. He answered that 1400 rounds off to 1000, 1700 and 1500 round off to 2000. Was he right? Why?

If the number in the hundreds' place is 5 or more, in rounding to the nearest multiple of one thousand, what do you do with the number in the thousands' place?

If the number in the hundreds' place is less than 5, what happens? (The number in the thousands' place remains the same.)

Round off these numbers to the nearest multiple of one thousand:

a) 455
b) 1,100
c) 8,545
d) 110,905
e) 9,998
f) 67,496

Do you see a pattern in this?

If the number in the thousands' place is 5 or more would you increase the digit in the 10,000 place by one? What would you do if the number in the thousands' place is less than 5?

Round these numbers to the nearest multiple of 10,000:

a)	15,000	d)	184,000
b)	11,111	e)	7,775,600
c)	9,200		

If you use place value of the digits in a numeral, it may help you in rounding off a number. Look at these examples.

a) Round of 3,495,000 to the nearest multiple of 1,000,000. Think of the number as 3 million + 495,000. We want to know if it should be rounded off to 3 million or 4 million. We look at 495,000 and see that it is less than one-half of 1,000,000. This tells us to round off the number to 3 million, 3,000,000.

b) Round off 7,775,600 to the nearest multiple of 100,000. Think of the number as 77 hundred thousands + 75,600. We want to know if it should be rounded off to 78 hundred thousands or to 77 hundred thousands. We look at 75,600 and see that it is more than one-half of 100,000. This tells us to round off the number to 78 hundred thousands, 7800,000 or 7,800,000.

These examples help us see that just one digit is important in rounding off a number.

In rounding off to the nearest multiple of one million, the hundred thousands' digit is important. If it is 5 or greater than 5, then the millions' digit is increased by 1. But if the hundred thousands' digit is less than 5, then the millions' digit is not changed.

In rounding off to the nearest multiple of one hundred thousand, the ten thousands' digit is important. If it is 5 or greater than 5, then the hundred thousands' digit is increased by 1. But if the ten thousands' digit is less than 5, then the hundred thousands' digit is not changed.

Does this help to see that in rounding off to the nearest 10,000 that the thousands' digit is important? Which digit is important in rounding off to the nearest 1000? the nearest 100? the nearest 10?

Copy each of these numbers. Draw a circle around the important digit. Then round off to the nearest multiple of one hundred thousand:

a) 749,000

b) 750,000

Round off to the nearest multiple of one million after circling the important digit.

c) 3,495,000

d) 3,675,112

e) 9,500,000

Now suppose you wished to round off to the nearest multiple of 5 or 50 or 500, how would you go about doing it?

We will use a number line again to help us in our thinking.

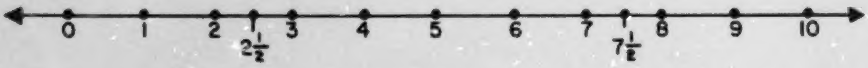


Figure 5

What is the mid-point between 0 and 5?

What is the mid-point between 5 and 10?

If we are rounding off to the nearest multiple of five, which numbers on the number line would be rounded to five? Would $2\frac{1}{2}$ be rounded to 5? Would $7\frac{1}{2}$ be rounded to 5? What would $7\frac{1}{2}$ be rounded to in groups of five? What would 1 and 2 be rounded to in groups of five?

Round off to the nearest multiple of 5:

a)	4	c)	7	e)	12
b)	9	d)	16	f)	342

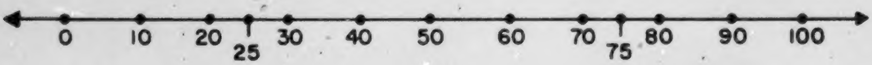


Figure 6

On the number line above what point is mid-way between 0 and 50?

What point is mid-way between 50 and 100?

What numbers could be rounded off to one multiple of 50?

Would 25 be rounded off to 1 multiple of 50?

If 75 were rounded off to the nearest multiple of 50, what would be the rounded number?

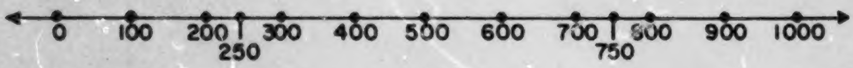


Figure 7

On this number line what point is midway between 0 and 500? Between 500 and 1000? What numbers can be rounded off to 500? Would 250 be rounded off to 500? Round off 750 to the nearest multiple of 500.

What is the pattern going to be if we round off to the nearest multiple of five, 50, 500, or 500,000? In rounding off to the nearest multiple of 500, the important digits in the following numbers are circled: 1325; 23,726; 91,304; 126,253. The rounded numbers for these are: 1300; 23,500; 91,500; 126,000.

Round off these numbers to the nearest multiple of 500,000.

a) 140,000
b) 300,000
c) 1,600,000
d) 250,000
e) 750,000

In locating points on a graph it would not be reasonable to graph ordered pairs expressed by exact numbers when the numbers are large. It would be very difficult to choose a scale in which we could show 40,000, 50,000, and 49,876 on a number line and show a difference between the last two numbers that we could use. Therefore, in graphing data using very large numbers we estimate or round off the numbers to fit the most convenient scale.

Exercise Set 2

1. Round each of these numbers to the nearest multiple of 10.

   a) 14
   b) 15
   c) 20
   d) 555
   e) 199
   f) 3999

2. Round each of these numbers to the nearest multiple of one hundred.

   a) 65 b) 102 c) 149 d) 851 e) 990

3. Round each of these numbers to the nearest multiple of one thousand.

   a) 8,052
   b) 3,716
   c) 9,499
   d) 9,500
   e) 10,050

4. Write each of these numbers rounded to the nearest multiple of ten thousand, the nearest multiple of one hundred thousand, and the nearest multiple of one million.

   a) 55,600
   b) 585,500
   c) 1,098,760
   d) 30,500,001
   e) 9,615,847,100
   f) 105,105,105

5. Round the following numbers to the nearest multiple of five hundred, the nearest multiple of five thousand, the nearest multiple of five hundred thousand.

   a) 2649 b) 864,492 c) 7,048,501

6. The population in 1960 of some of the states is given below. Their rank, according to population, is given for 1950 and for 1960. Make a table to show this information. Use four columns: State, 1960 Population, 1960 Rank, 1950 Rank. List the state with the largest population first and list the others in order of decreasing population.

Alabama	3,266,740;	1950 rank 17;	1960 rank 19
Arizona	1,302,161;	1950 rank 38;	1960 rank 35
California	15,717,204;	1950 rank 2;	1960 rank 2
Connecticut	2,535,234;	1950 rank 28;	1960 rank 25
Florida	4,951,560;	1950 rank 20;	1960 rank 10
Hawaii	632,772;	1950 rank 46;	1960 rank 44
Illinois	10,081,158;	1950 rank 4;	1960 rank 4
Iowa	2,757,537;	1950 rank 22;	1960 rank 24
Kentucky	3,038,156;	1950 rank 19;	1960 rank 22
Maine	969,265;	1950 rank 35;	1960 rank 36
Massachusetts	5,148,578;	1950 rank 9;	1960 rank 9
Minnesota	3,413,864;	1950 rank 18;	1960 rank 18
Missouri	4,319,813;	1950 rank 11;	1960 rank 13
Nebraska	1,411,330;	1950 rank 33;	1960 rank 34
New Hampshire	606,921;	1950 rank 45;	1960 rank 46
New Mexico	951,023;	1950 rank 40;	1960 rank 37
New York	16,782,304;	1950 rank 1;	1960 rank 1
North Carolina	4,556,155;	1950 rank 10;	1960 rank 12
Ohio	9,706,397;	1950 rank 5;	1960 rank 5
Pennsylvania	11,319,366;	1950 rank 3;	1960 rank 3
Texas	9,579,677;	1950 rank 6;	1960 rank 6
Virginia	3,966,949;	1950 rank 15;	1960 rank 14
Washington	2,853,214;	1950 rank 23;	1960 rank 23
Wyoming	330,066;	1950 rank 49;	1960 rank 49

7. Round to the nearest multiple of 500,000 the population given in exercise 6.

   a) What three states have rounded populations of one-half million?

   b) What three states have rounded populations of three million?

   c) Which state has a rounded population of $2\frac{1}{2}$ million?

   d) In rounding to the nearest multiple of 500,000, which states would appear to have larger populations than shown by the numbers in exercise 6? Which seem to have smaller populations?

BRAINTWISTERS

1. Make a table of the population of the fifty states, using the final 1960 census figures. Arrange in order of population as you did in the preceding exercise 6.

2. Obtain data of interest to you. Show how it can be placed in a table to make it easy to understand.

Exploration

In the chapter on Coordinates you learned that you could draw a picture of a set of ordered pairs of numbers by associating each ordered pair with a point of the plane. The point was called the "graph" of the ordered pair. The tables in Chapter 8 contain examples of ordered pairs, although both members of the pair are not necessarily numbers. For example, in the vote on a class pet, we had the ordered pairs (hamster, 9 votes), (guinea pig, 4 votes), etc. Thus we should be able to set up some type of a coordinate system and locate points to correspond to our data. These points are called a graph of the data.

In graphing data, we usually use only one quadrant and the axes are called reference lines. It is not necessary to use the same scale on each reference line. For example, one reference line might represent years and the other reference line might represent population. Your reference line for years might only have to indicate numbers to a little beyond 1960, but your reference line for population might need to indicate numbers as high as 2,905,600,000. A graph has a title which tells what the graph shows. Graph paper is often used for constructing graphs.

1. The results of the vote for a class pet may be shown on a graph. But, instead of numbers, names of the pets would be listed on the vertical line. Five evenly-spaced points on the line are chosen and each one labeled with the name of one of the pets. It is easy to see that we need to use only 5 spaces because there are just 5 pets. If we wanted to have a graph that was "spaced out" more, we could have skipped one or two lines between the names of the pets. The pets' names would then be evenly spaced but would be farther apart. The largest vote for any pet was 9 so we choose 9 evenly-spaced points on the horizontal reference line and label this line "Number of Votes."

2. Let's suppose that all the children in the school were voting for a pet. Then one animal might get as many as 360 votes. We do not have 360 of the evenly-spaced vertical lines. Therefore, we cannot let each line stand for just one vote. We would need to let each line stand for several votes.

   If we chose to let each line stand for 5 votes, then we would need 72 lines because there are 72 fives in 360. Are there 72 vertical lines on your graph paper? Probably not! Then let's let each line represent 25 votes, for example. We would need 15 lines because 14 twenty-fives would show 350 votes and we want to show more votes than that.

a) If we let each line stand for 50 votes, how many lines would we need?

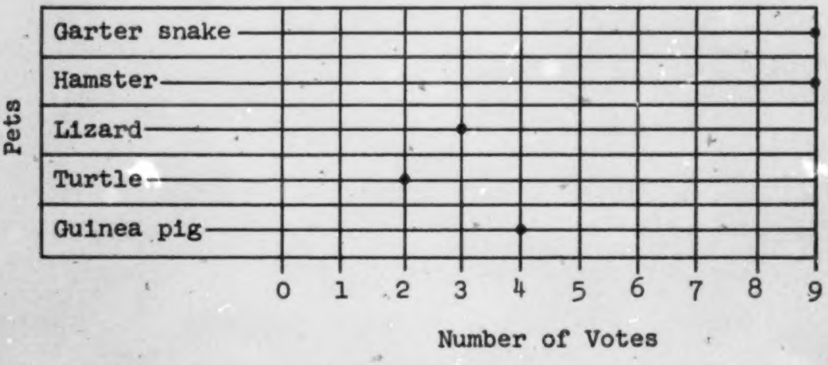
b) If we let each line stand for 10 votes, how many lines would we need?

Before we can draw a graph, we must look at the facts we want to graph. We will need to decide how to show these facts on a graph. We may need to let the lines on the graph paper stand for more than just one fact, as we explained above.

Now locate the points corresponding to the vote for each pet. Our graph might look like this:

Graph of

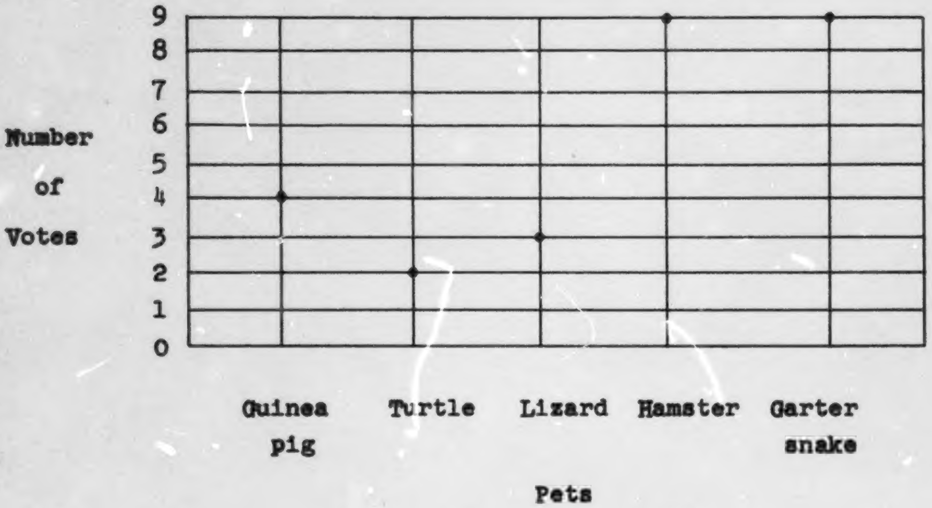
Vote for Class Pet



3. Could we have used the vertical axis to show the number of votes? Our graph then might look something like this:

Graph of

Vote for Class Pet



4. Does this graph tell the same story? From the graph, tell which pets received 0 votes, 4 votes, 9 votes.

5. In some graphs the points are connected by line segments. These line segments show approximately where other points would be if additional data were obtained. In this graph, line segments would have no meaning and should not be drawn. For example, there isn't any animal halfway between guinea pig and turtle.

LINE SEGMENT GRAPHS

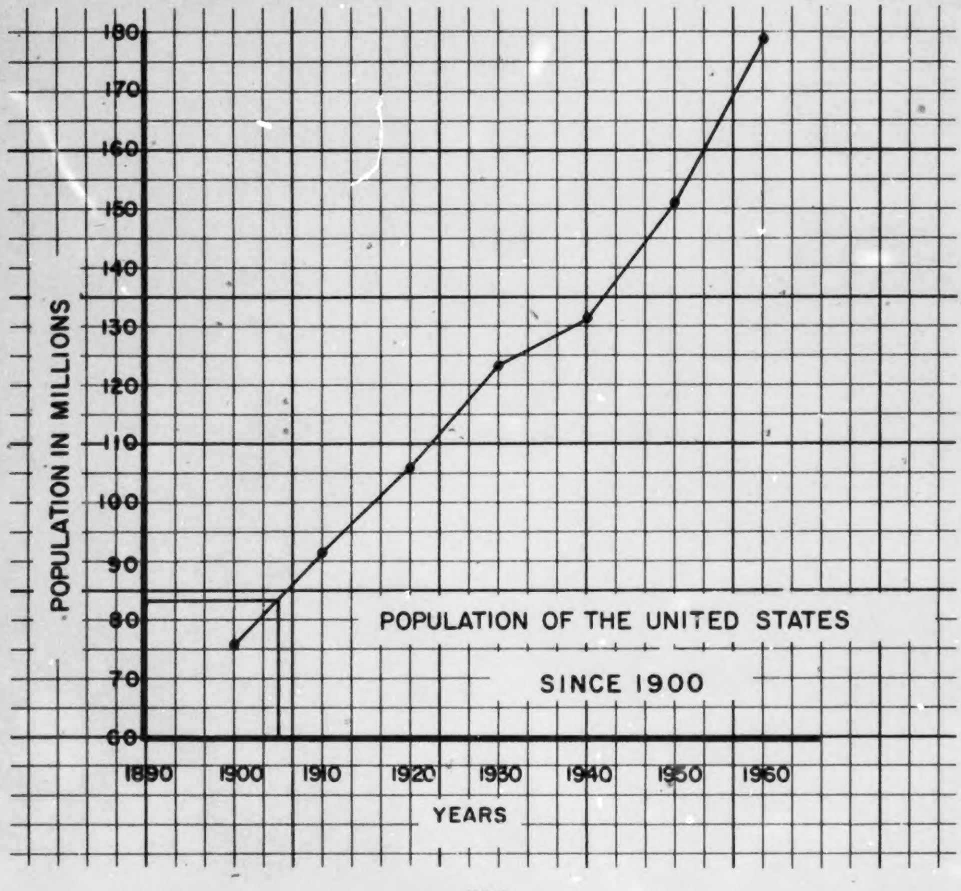
Exploration

The population of the United States for each ten year period since 1900 is shown in the following table. The numbers in the Population column have been rounded to the nearest million.

Census Years	Population in Millions
1900	76
1910	92
1920	106
1930	123
1940	132
1950	151
1960	179

1. The graph of this data is shown following exercise 2 of this set of exercises. The horizontal reference line is usually used as the reference line for time as measured in hours, months, years, or any time interval. Since there are seven 10-year periods represented, we choose 7 equal spaces that use most of the horizontal line segment. The vertical line is used as the reference line for the population. The population difference between 1900 and 1960 is 103 million. (The difference between 179 million and 76 million.) We can let each space on the vertical line represent 10 million in population. We choose 12 equally spaced dots on the vertical line. The bottom dot represents 70 million, the next dot 80 million, etc. The ordered pairs are graphed.

2. It is meaningful in this example to connect the points with line segments. The line segments are drawn. We will call this graph a line segment graph. In this case, the points of the line segments give us estimates of the population between census years. For example, the population in 1905 can be estimated. Find the point on the horizontal axis which represents 1905. There is a vertical line at this point. This vertical line intersects the line segment graph. Follow a horizontal line from this point of intersection to the vertical axis on which the population numbers are shown and estimate the numbers on this axis as carefully as you can.

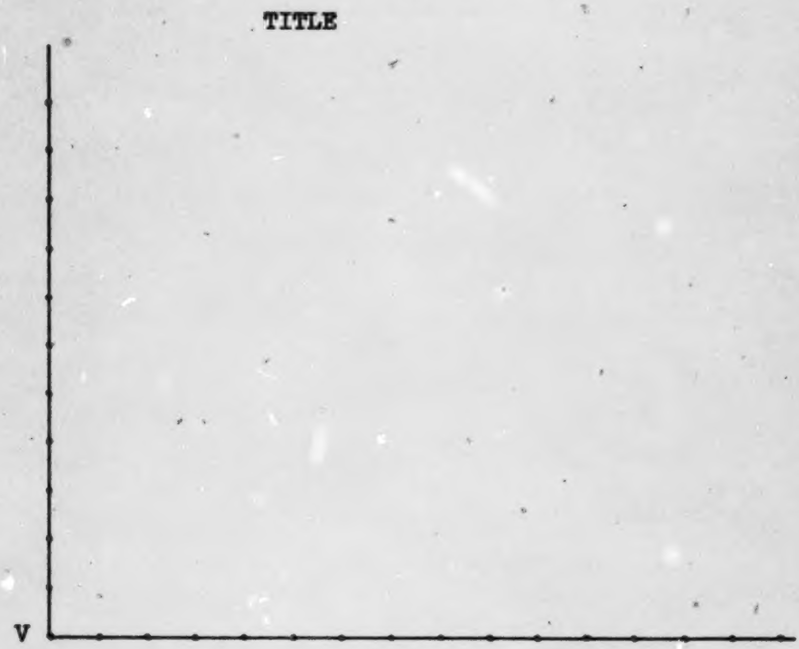


3. From the graph, estimate the population in 1935 and in 1955. Could you estimate the population in 1965?

4. Why could we draw a line segment graph for the population data but not for the data showing the vote on the class pet?

A line segment graph can be drawn only when points of the graph between the endpoints of each line segment represent an estimate of some ordered pair of numbers that could be considered part of the data. Suppose we had taken the population count every year instead of every ten years. Then we could have plotted more points. Our line segment is just a guess at where these points would be located. Thus, you should not use a broken-line graph when you have objects like hamster and turtle represented by points on one of the reference lines. A type of graph that could be used to show the vote on the hamster and the turtle is a bar graph which will be discussed in the next section.

Suppose you are preparing to graph the pairs in the columns (a) and (b) that are shown below. Suppose that you have horizontal and vertical reference lines as shown in the drawing. There is space for the title and numbering of the reference lines. There are 15 spaces on the horizontal reference line and 11 spaces on the vertical reference line.



(a)		(b)	
6,	88	1790	7,400,000
12,	99	1800	9,600,000
18,	110	1810	11,800,000
24,	121	1820	14,000,000
30,	132	1830	16,200,000

1. This part is about column (a). Label the first numbers of the pairs on the horizontal reference line. Begin numbering the vertical scale at 65 and the horizontal scale at 0 starting at the point V. Fill the blanks.

a) Choose ______ horizontal spaces to represent 1 and ______ vertical spaces to represent 1.

b) The farthest labeled point to the right on the horizontal scale is labeled ______. It is ______ spaces from V.

c) The highest labeled point on the vertical axis is labeled with ______. It is ______ spaces from V.

This part is about column (b): Label the first numbers of the pairs on the horizontal reference line. Begin numbering the vertical scale at 72 and the horizontal scale at 1790, starting at the point V.

d) Choose ______ horizontal spaces to represent 10 and three vertical spaces to represent ______.

e) The farthest labeled point to the right on the horizontal scale is labeled ______. It is ______ spaces from V.

f) The highest labeled point on the vertical axis is labeled ______. It is ______ spaces from V.

2. On a vertical reference line you have to include numbers between 265 and 530. You have 31 points that you may number There are 30 spaces.

What number would you let each space represent?

How would you number the first point and the last point?

3. On a horizontal reference line you have to include numbers from 1850 to 4250. You have 12 spaces on the reference line.

What would you let each space represent?

What numbers would you use to label the first and last points?

Exercise Set 3

1. The boxes of mint cookies sold by the Girl Scout troops were: Troop 1, 48 boxes; Troop 2, 34 boxes; Troop 3, 72 boxes; Troop 4, 25 boxes; Troop 5, 75 boxes; Troop 6, 51 boxes; Troop 7, 132 boxes; Troop 8, 82 boxes. May a line segment graph be used?

2. Anna weighed 6 pounds at birth, 12 pounds at 6 months, 19 pounds at 12 months, 28 pounds at 18 months, and 34 pounds at 24 months.

   Show her rate of growth using a line segment graph.

   a) Does the graph you made show Anna's growth to be the same for each six month period?

   b) During which period is her gain in weight the greatest?

3. Max Q. Farmer raised a calf. At the end of a two-year period he had recorded the following information:

   Weight at birth: 70 pounds
   Weight at age 6 months: 400 pounds
   Weight at age 12 months: 600 pounds
   Weight at age 18 months: 1100 pounds
   Weight at age 24 months: 1400 pounds

   Graph these facts using a line segment graph.

   a) During which period in the calf's life was the greatest gain in weight shown?

   b) How does the gain in weight of the first six months and the last six months compare?

   c) How are the two graphs in exercises 2 and 3 similar?

4. Gasoline costs 30¢ a gallon. Complete the table:

Gasoline	1	2	3	4	5	6	7	8	9	10
Price	30¢	60¢	?	?	?	?	?	?	?	?

Draw a line segment graph to show this information.

BRAINTWISTER

The cost of mailing a first-class letter is 5 cents if the weight of the letter is one ounce or less. The cost is increased by 5 cents for each additional ounce or fractional part of an ounce. For example, it would cost 10 cents to mail a letter weighing $1\frac{1}{3}$ ounces and it would also cost 10 cents to mail a letter weighing 2 ounces. It would cost 15 cents to mail a letter weighing $2\frac{1}{5}$ ounces.

Draw a graph showing the postal rate for first-class letters up to 5 ounces.

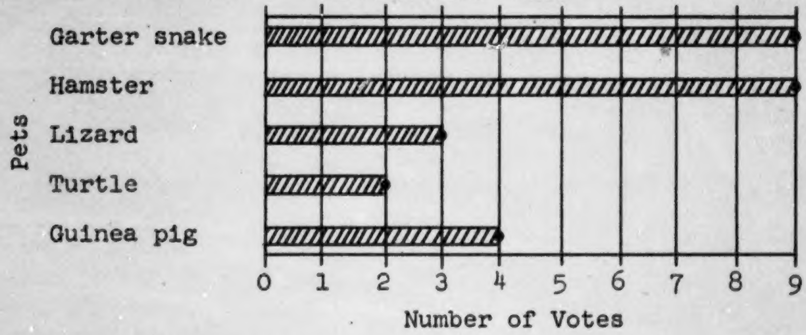
Read from your graph the cost of mailing a letter weighing $4\frac{1}{2}$ ounces.

BAR GRAPHS

Exploration

In the last section we found that the results of the election for a class pet could not be shown on a broken-line graph. The entire graph is just the five points, and it is not possible to get any additional points. However, the five dots are difficult to see, so we can draw bars from the reference line for the pets to the points like this:

Bar Graph
of
Vote for Class Pet

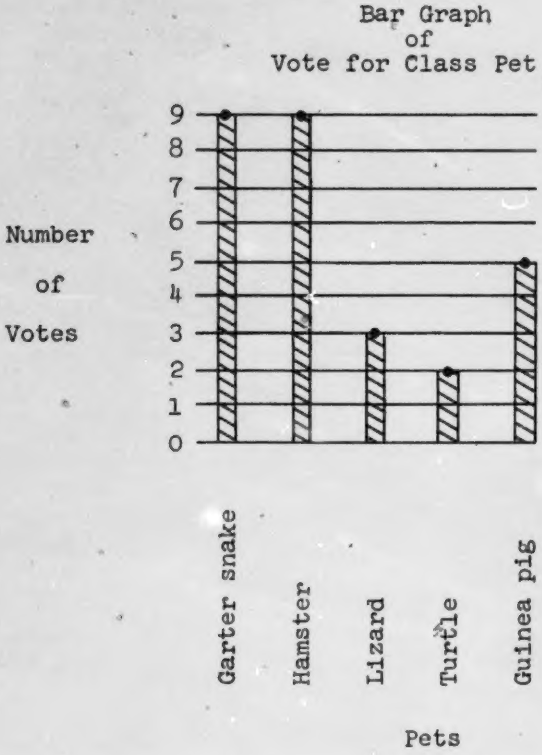


This type of graph is called a bar graph. Is a bar graph easier to read than just a graph of an isolated set of points?

A bar graph is used to compare things, such as the vote for each pet.

The bars should be the same width. The spaces between the bars should be the same width, but not necessarily the same as the width of the bars.

The bars on a bar graph may be vertical rather than horizontal. To make this type of graph for the vote for a class pet, place the names of the pets along the horizontal reference line. Why? The graph might look like this:



Exercise Set 4

1. A table of newspaper sales is given below. Make a bar graph showing the total papers sold by George, Harry, Jim, and John for one week.

TABLE 3

Newspaper Sales for One Week

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
George	3	10	18	30	52	70	98
Harry	28	29	29	30	32	60	75
Jim	15	21	47	47	47	50	54
John	36	38	38	40	42	44	45

2. Make a bar graph showing the number of papers sold by George during each day of the week.

3. Make a bar graph showing the number of papers sold by each boy on Saturday.

4. The following American ski-jumping records have been made. Graph this information, using a bar graph.

1887	Mikkel Hemmestvedt	37 feet
1905	Julius Kulstad	97 feet
1910	Oscar Gunderson	138 feet
1915	Ragnar Omtvedt	192 feet
1920	Lars Haugen	214 feet
1932	Glenn Armstrong	224 feet
1940	Torger Tokle	228 feet
1950	Art Devlin	307 feet
1951	Austen Samuelstuen	316 feet

The record of 316 feet is the present American record and still stands. Longer jumps have been made in Europe.

5. Draw a bar graph to show the sale of boxes of mint cookies by these Girl Scout troops.

Troop 1 - 48 boxes
Troop 2 - 34 boxes
Troop 3 - 72 boxes
Troop 4 - 25 boxes
Troop 5 - 75 boxes
Troop 6 - 51 boxes
Troop 7 - 132 boxes
Troop 8 - 82 boxes

6. The Empire State Building in New York City is the world's tallest building. Its height is 1,472 feet. The heights of some other structures are: Eiffel Tower, 984 feet; Washington Monument, 555 feet; Pyramid of Cheops, 480 feet; Leaning Tower of Pisa, 179 feet.

Draw a bar graph to illustrate this data.

7. The ages of the Presidents at the time they took office were: Washington, 57; J. Adams, 61; Jefferson, 57; Madison, 57; Monroe, 58; J. Q. Adams, 57; Jackson, 61; Van Buren, 54; W. H. Harrison, 68; Tyler, 51; Polk, 49; Taylor, 64; Fillmore, 60; Pierce, 48; Buchanan, 65; Lincoln, 52; Johnson, 56; Grant, 46; Hayes, 54; Garfield, 49; Arthur, 50; Cleveland, 47; B. Harrison, 55; Cleveland, 55; McKinley, 54; T. Roosevelt, 42; Taft, 51; Wilson, 56; Harding, 55; Coolidge, 51; Hoover, 54; F. D. Roosevelt, 51; Truman, 60; Eisenhower, 62; Kennedy, 42.

Make a bar graph to show this data.

CIRCLE GRAPHS

Exploration

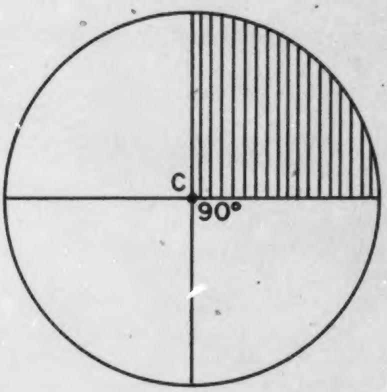
1. What is a circle?

2. What is a diameter?

3. Into how many half-circular regions does a diameter divide a circle and its interior?

4. Draw a circle using your compass. Draw a diameter of this circle as in the picture. Label the center of the circle C.

5. Use your protractor to draw another diameter. Draw it so that the angle between the diameters is a 90° angle. How many right angles are formed? What is the measure, in degrees, of a right angle? Can you find four right angles whose vertices are at C?



6. Shade one of the regions as in the picture. Is this shaded region one-fourth of the circular region?

   Suppose an angle of one degree is drawn in the shaded region with its vertex at the center of the circle. This angle makes in the shaded region a region shaped like a very small piece of pie. There are 90 of these small pie-shaped regions in the entire shaded region. Why?

7. How many of these small pie-shaped regions would there be in the whole circular region? In half of a circular region? In one-eighth of a circular region?

We can think, then, of a circular region separated into 360 regions, each shaped like a small piece of pie.

8. What would be the measure, in degrees, of an angle which forms: a) $\frac{1}{6}$ of a circular region? b) $\frac{3}{10}$ of a circular region? c) $\frac{5}{12}$ of a circular region?

9. How would you find how many of these small pie-shaped regions there would be in any fractional part of a circular region?

10. Three boys, Dave, Peter, and Ron, planned to go on a camping trip. They decided to work and earn as much money as possible. Then they would put all the money together. Dave earned $4, Peter earned $6, and Ron earned $8. This made a total of $18.

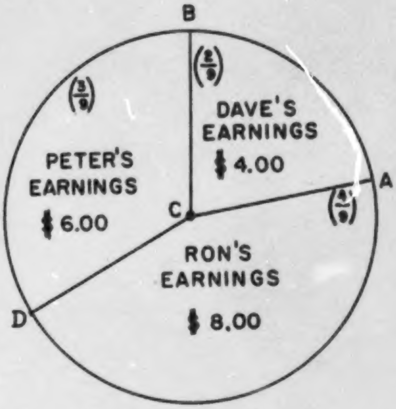
a) What part of the $18 did Dave earn?
b) What part of the $18 did Peter earn?
c) What part of the $18 did Ron earn?

Dave earned $\frac{4}{18}$ or $\frac{2}{9}$ of the total money, Peter earned $\frac{6}{18}$ or $\frac{1}{3}$ of the money, and Ron earned $\frac{8}{18}$ or $\frac{4}{9}$ of the money.

11. Suppose the boys wished to show by a circle graph what part of the $18 each gave. How could they do it? They would probably follow these steps:

a) With the compass, draw a circle of a convenient size. We will show each boy's share of the earnings by a "piece of pie." The entire "pie" represents the total earnings.

b) To show Dave's earnings we can multiply $\frac{2}{9}$ by 360, since there are 360 pie-shaped regions in a circular region. ($\frac{2}{9} \times 360 = \frac{720}{9} = 80$)



c) With your protractor draw an angle of 80° with vertex at the center of the circle. You have divided the whole pie into two regions. Which region represents Dave's earnings?

(To show Peter's earnings we multiply $\frac{1}{3}$ by 360. $\frac{1}{3} \times 360 = 120$. With a protractor measure an angle of 120°. The angle should have a ray common with the previous angle.)

12. Ron's earnings should be represented by the remaining piece of pie. Angle ACD is an angle at the center of the circle equal to $\frac{4}{9}$ of 360° or 160°. Is it?

Let us do a little checking:

a) How much is the sum: $\frac{2}{9} + \frac{3}{9} + \frac{4}{9}$?

b) How much is the sum of \$4.00 and \$6.00 and \$8.00?

c) How much is the sum of 80° and 120° and 160° ?

The sum of the fractional parts of the circle should equal 1.

The amount of money given by the boys should equal \$18.00.

The sum of the measures of all the angles should equal 360.

13. Does our circle graph show that these three statements a), b), and c) in exercise 12 are correct?

Exercise Set 5

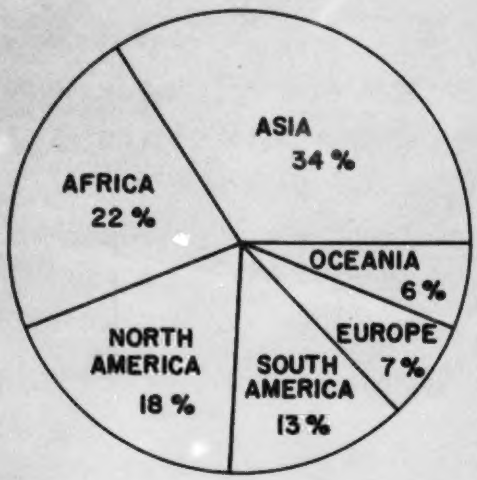
1. There are 30 pupils in the sixth grade at Lincoln School. There are 14 of these pupils that live less than one mile from school, 10 that live more than one mile and less than two miles, and 6 that live more than two miles.

   Represent these data on a circle graph.

2. Mary gets an allowance of one dollar a week. She spends 50¢ each week for entertainment (such as movies). She saves 20¢ in her savings bank each week and gives 10¢ to the church. She spends 20¢ each week for school supplies.

   Draw a circle graph which shows the percent of her allowance that Mary uses for each purpose.

3. Below is a circle graph. Study it and then answer the question which follow it.



AREA OF CONTINENTS AS PER CENT OF WORLD LAND AREA

a) Europe and Asia combined make up what per cent of the world land area?

b) Is the combined area of North and South America as great as that of Asia?

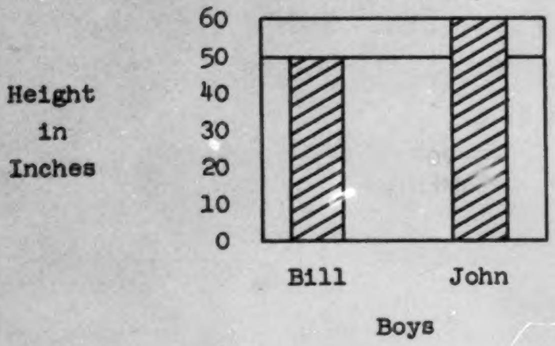
c) Is the land area of North America more or less than $\frac{1}{5}$ of the world land area?

d) Should the measure of the angle which determines the region representing Asia's area be more or less than 120?

PICTOGRAPHS

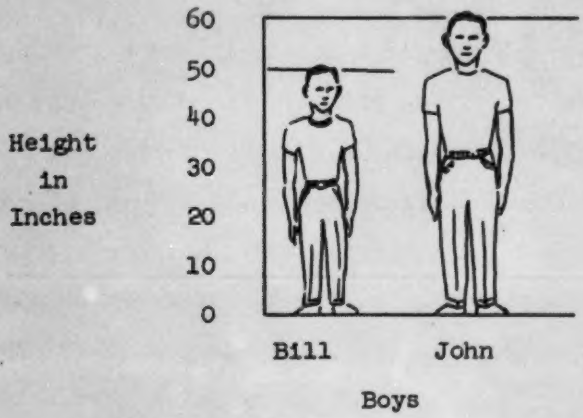
Exploration

John was 60 inches tall and Bill was 50 inches tall. John decided to draw a bar graph to impress Bill with how much taller he was. The graph looked like this:



Heights of Two Boys

John decided this graph wouldn't impress Bill very much so he decided to draw pictures of himself and Bill to replace the bars. Now the graph looked like this:



Heights of Two Boys

1. Does the graph make the difference in heights seem greater than it actually is? Why?

When a graph shows pictures of the objects represented by the data, as in the last example, it is called a pictograph. Some pictographs may give an unfair picture of the data being presented. For example, the pictograph of the heights of Bill and John shows John is not only taller, but is also larger! That is, his shoulders are wider, his head is larger, and so on.

2. In 1951, the number of telephones in use in the United States was 66,645,000 and in Europe the number was 37,593,000. Round these figures off to the nearest million and draw a bar graph. Then draw a pictograph by drawing two telephones, each one as tall as the length of the corresponding bar of the bar graph. Does this pictograph represent a fair picture of the data?

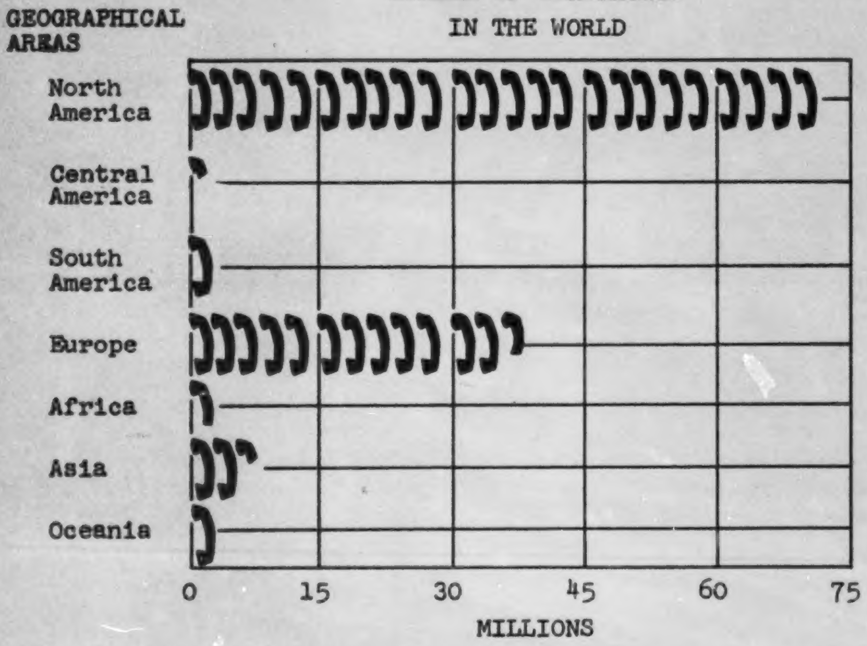
3. Another way of drawing a pictograph of the number of telephones in various parts of the world would be to draw one telephone for each 3,000,000 telephones. Africa, with about 2,000,000 telephones would have to be represented by a picture of $\frac{2}{3}$ of a telephone. This pictograph is shown on the following sheet. Do you get a better understanding of the telephones in the United States and Europe from the pictograph of exercise 2 or exercise 3?

World Telephone Statistics

Area	Total Number of Telephones
North America	71,803,700
Central America	910,800
South America	2,999,600
Europe	37,593,900
Africa	1,768,600
Asia	6,855,500
Oceania	2,867,900

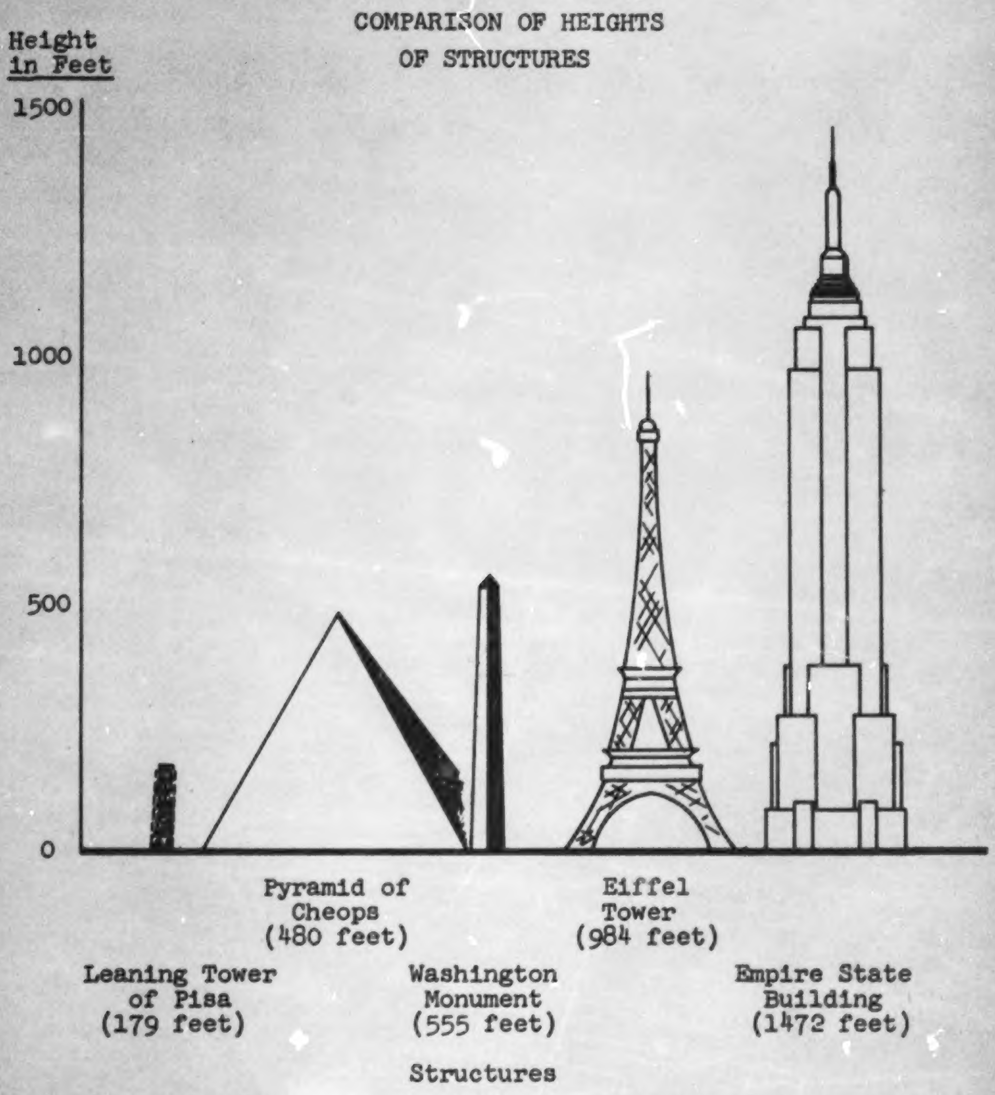
Round these numbers to the nearest one million. If we draw one picture for each 3 million telephones, how many pictures of telephones will we draw for each country?

NUMBER OF TELEPHONES
IN THE WORLD



(Each symbol represents 3,000,000 telephones.)

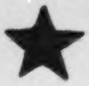
5. **A pictograph for Problem 6 of Exercise Set 5 on the height of the Empire State Building and other structures is shown below.**



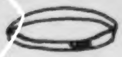
Do you like the pictograph better than the bar graph?

Exercise Set 6

1. On Major Cassidy's Dude Ranch there are 22 horses, 18 milk cows, 4 dogs, and 12 cowboys. Let us choose a symbol to represent horses, another to represent cows, and still other symbols for dogs and cowboys. The following symbols will be used:

A star represents horses 

A set of horns represents cows

A dog collar represents dogs 

A stick-man represents cowboys

Make a pictograph of this data. Let each symbol represent a group of 4.

2. Draw a pictograph to compare the population of the United States in 1900, 1930, and 1960. The 1900 population was 76 million, the 1930 population was 123 million, and the 1960 population was 179 million.

3. The highest altitude in each of these states is given. Round off each of the numbers to the nearest multiple of 100, then draw a bar graph. Put the names of the states along the horizontal reference line.

Alaska	20,320 feet
Arizona	12,670 feet
Colorado	14,431 feet
Hawaii	13,796 feet
Maine	5,268 feet
Massachusetts	3,491 feet
New York	5,344 feet
Tennessee	6,642 feet
Washington	14,410 feet
Wisconsin	1,941 feet

BRAINTWISTER

Using the 1960 census figures, draw a rough map of western United States. Let the size of each state depend upon its population. That is, the states which have the largest population should be drawn larger than states which have smaller populations. Use Washington, California, Oregon, Montana, Idaho, Nevada, Wyoming, Arizona, New Mexico, Texas, and Colorado.

MEASURES OF CENTRAL TENDENCY

Exploration

Once the data have been organized, the next problem is to try to find a number which will describe the data and help us understand it. A number which tells us something about how near the number pairs are to some central number pair is called a measure of central tendency, or an average. There are several different types of averages.

1. Here are the scores of Bill, John, and Mike, on five tests:

	Test 1	Test 2	Test 3	Test 4	Test 5
John	80	75	80	100	80
Bill	40	80	60	80	80
Mike	25	80	30	81	82

a) For each boy, arrange the scores from smallest to largest.

b) How would you find the "average" of John's scores?

Probably you found the "average" by adding the scores together and dividing by the number of scores, like this:

$$\frac{75 + 80 + 80 + 80 + 100}{5} = 83$$

Are many of the scores in the table "near" 83? The scores of 75, 80, 81, 82 are not far from 83. This type of average is called the *arithmetic mean*. The word "arithmetic" in "arithmetic mean" is not pronounced the way you usually pronounce the word. Look in the dictionary for the correct pronunciation. The *arithmetic mean* is an average computed by using *arithmetic* (addition and division are used). Thus you should be able to remember the name of this average.

c) Find the arithmetic mean of Bill's scores and of Mike's scores.

d) There may be some other "average" that would make Bill's grades look better.

The number that occurs most frequently in a set of numbers is another type of average called the mode. We think of the most popular type of dress or hat as this year's mode or style; hence, the choice of the word mode. If no item occurs more than once, there is no mode.

e) What is the mode of Bill's scores? Is it better than the arithmetic mean of his scores?

f) What is the mode of John's scores? Is it better than the arithmetic mean of his scores?

g) There is still a third type of average that might be useful in comparing the scores.

When the numbers of a set are arranged in order of increasing or decreasing size, the median is the number that is in the middle. There are just as many numbers below the median as above. The word "median" means "middle." By associating these two words you should be able to remember that the median score is the middle score.

h) What is the median of Mike's scores? Is it better than the arithmetic mean of his scores?

i) Make a table and list the arithmetic mean, median, and mode for the scores of John, Bill, and Mike.

j) Which type of average seems to be the fairest for describing the various sets of test scores?

2. In example 4 of the Exploration on Organizing Data near the first page of this chapter you were asked to make a table showing the number of boxes of cookies sold by eight Girl Scout troops. Your table probably looked like this:

TABLE 4

Cookie Sales
by
Girl Scout Troops

Troop	Mint	Chocolate	Vanilla
1	48	63	35
2	34	27	30
3	72	51	40
4	25	14	12
5	75	39	51
6	51	62	37
7	132	98	99
8	82	98	76

The arithmetic mean of the numbers of boxes of mint cookies sold by the different troops is

$$\frac{48 + 34 + 72 + 25 + 75 + 51 + 132 + 82}{8} = 64\frac{7}{8}$$

a) Which three troop's sales of mint cookies were nearest the arithmetic mean?

b) Arrange the numbers of boxes of mint cookies sold from the smallest number to the largest number.

To find the median number sold, we have to find the median of eight numbers. Because 8 is an even number, there isn't any one middle number. So consider the two middle numbers. These are 51 and 72. Any number between 51 and 72 would be a number such that four sales were less (25, 34, 48, 51) and four greater (72, 75, 82, 132). Usually, in cases like this, the arithmetic mean of the two middle numbers is chosen as the median. The arithmetic mean of 51 and 72 is

$$\frac{51 + 72}{2}, \text{ or } \frac{123}{2}, \text{ or } 61\frac{1}{2} .$$

The median is $61\frac{1}{2}$.

c) Which troop's sales of mint cookies were nearest the median?

d) Find the arithmetic mean of the sales of chocolate cookies and of vanilla cookies.

e) Which two troop's sales of chocolate cookies were nearest the arithmetic mean?

f) Find the median of the sales of chocolate cookies and vanilla cookies.

g) Which two troop's sales were nearest the median?

Exercise Set 7

1. Farmer Jones can grow a good garden if about four inches of rain falls each month during the growing season from May until October. The rainfall during these months last year was:

May	1 inch
June	0 inches
July	0 inches
August	1 inch
September	10 inches
October	12 inches

Compute the arithmetic mean and the median. Is the average monthly rainfall 4 inches, using one of these averages?

Do you think Farmer Jones had a good garden? Why?

2. The principal announced that the average number of students in the fifth and sixth grades was 25. By average he meant the arithmetic mean. There was one class of each grade. There were 15 pupils in the fifth grade.

How many pupils were in the sixth grade?

3. The temperature in degrees at noon on the first day of each month in the town of York were recorded as follows:

January	12	July	105
February	12	August	105
March	32	September	62
April	55	October	45
May	76	November	17
June	105	December	12

Find the arithmetic mean and the median. Do these averages indicate that York would have a pleasant year-round temperature? Find the mode. This distribution has two modes so list both of them. Now do you think York has a pleasant year-round temperature?

4. Table 3 near the front of this chapter shows the newspaper sales of four newsboys.

a) Find the arithmetic mean of the weekly sales for George, Harry, Jim, and John.

b) Find the median of the weekly sales for each of the newsboys.

c) Which average shows better how busy the newsboys usually are?

5. From the data on the ages of the presidents at the time they took office (Exercise Set 5, exercise 7), find the arithmetic mean and median.

Chapter 9

SETS AND CIRCLES

RECOGNIZING SETS

If you saw a great many birds flying overhead and wanted to tell someone about what you saw, how would you tell them? Would you say "I saw a *lot* of birds" or "I saw a *flock* of birds" or "I saw a *group* of birds"? You would be more likely to say "I saw a *flock* of birds". But, if you saw bees instead of birds, you might say "I saw a *swarm* of bees", or "I saw a *bunch* of bees".

A man who has a big ranch in Texas may have many cattle and many horses. If he were to talk to you about these, he might say he had a "*herd* of cattle" and a "*drove* of horses."

You see, in these first two paragraphs we have used seven different words in the same way. These words are underlined. Is each a word used to refer to a collection of things? Now, you may remember that we decided to use just one word to speak about a collection of things. This word was the word *set*. If we do this we would say "A set of birds", "A set of bees", "A set of cattle", "A set of horses".

If someone tells you that she is thinking about a set of dishes, does this tell you very much about the dishes in the set? Do you know what color they are? How many are in the set? Are they cups, or saucers, or plates, or some of each of these? The answer to most of these questions is "No". So, when someone says "a set of dishes" she is not really telling you very much about them. But if she were to say "the dishes you see in the window at a certain china store", she would be talking about a certain set. You cannot know what dishes are meant by "a set of dishes". But you can know what dishes are meant by "the set of dishes you see in the window at a certain china store".

Do you know what boys are meant if we say "a set of boys"? Do you know what boys are meant if we say "the set of boys in your class whose names are Tom, Dick, or Harry"?

In each of the first five exercises below, write on your paper a mathematical sentence for the set. Notice this example first.

Example: The set of streets that cross at First Street and Main Street. S = {First Street, Main Street}.

1. The set of odd whole numbers which are less than 10.

T = {

2. The set of even numbers which are less than twenty and greater than 8.

E = {

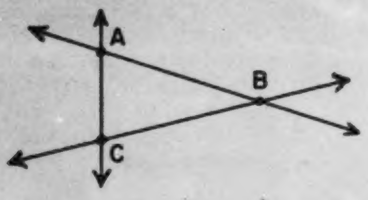
3. (a) The set of whole numbers which are factors of 30.

W = {

(b) The set of prime numbers which are factors of 30.

Q = {

4. The set of letters which label the points of intersection of the three line segments shown in this drawing.



P = {

5. The names of the states in the United States whose names begin with H.

Y = {

In some of the next five sets you are not told enough about the sets so you can write a mathematical sentence for them. Which ones are they?

6. Set of stamps.

7. Set of books on a certain shelf of the bookshelf in your classroom.

8. Set of letters in the first half of the alphabet.

9. Set of whole numbers less than 119 whose numerals have 1 for the first (leftmost) digit.

10. Set of cards.

Exercise Set 1

For each of the first five exercises below, write a mathematical sentence on your paper for each set.

1. The set of counting numbers less than 40 which are multiples of 5.

2. The set of prime numbers which are greater than 17 and less than 29.

3. The set of whole numbers which are factors of 60.

4. The set of whole numbers less than 480 which are multiples of 60.

5. The set of names of the states of the U.S.A. whose first letter is Z.

In some of the next six exercises you are not told enough about the sets so you can write a sentence for them. Which ones are they?

6. Set of letters.

7. Set of whole numbers which are greater than 10 and less than 11.

8. Set of points which are the vertices of a triangle.

9. Set of members of this set: {1, 4, +, Ͳ, £}.

10. Set of titles of the books that are on your teacher's desk.

11. Set of names of the presidents of the U.S.A. since 1900.

12. Describe in words the set { }.

13. Would the dishes your mother used last Thanksgiving and your father's best hammer form a set? Why?

14. Would the men in your town that are at least six feet tall and weigh no more than one pound form a set? Why?

15. Name the members of the set of

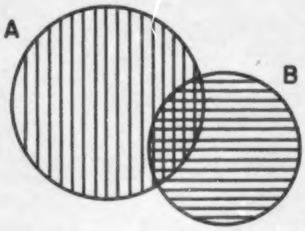
(a) red-haired boys in your class

(b) blue-eyed boys in your class

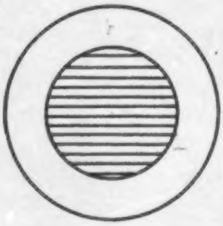
(c) red-haired blue-eyed boys in your class.

INTERSECTION OF SETS

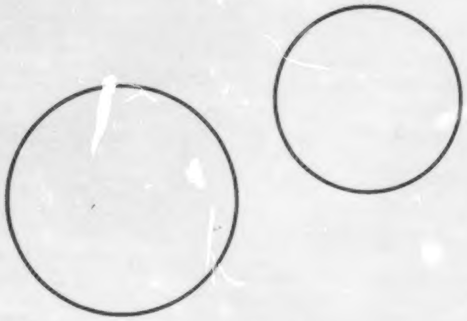
The two circular regions below are drawn so that they overlap. The circular region A is lined vertically. The circular region B is lined horizontally. The intersection is cross-hatched. The union of the interior of the two circular regions consists of three portions: The portion that is lined vertically, the portion that is lined horizontally, and the portion that is cross-hatched. Remember that union of two sets of points includes all the points that are points of either one of the sets or points of both of the sets.



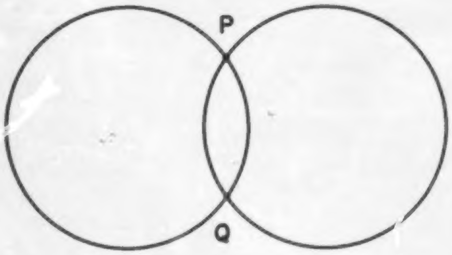
In the drawing below the interiors of the two circular regions intersect in the area that is shaded. Remember that the intersection of two sets is the set of elements that are in both sets. The members in the sets in this figure are the points of the two circular regions. The points of the smaller regions are in the larger region, so the intersection set is the set of points of the smaller region.



In the next drawing the union of the two circles consists of the two circles. Or you may say that the union of the sets of points on the two circles consists of all the points on the two circles. Draw the circles on your paper and color the union set with a colored crayon.



In the next drawing there are just two points that are on both circles. Name them P and Q. Are the points P and Q members of the intersection set of the interiors of the two circles? Are they members of the union set of the two circles? Are they the only members of the intersection set of the two circles?



Do you remember what we mean by the intersection of two sets? Perhaps an example will help recall it.

Look at these two sets:

$$A = \{c, a, n\}$$

$$B = \{b, a, t\}$$

The letter a is in both sets. We call this the intersection of sets A and B. We write this

$$A \cap B = \{a\}$$

The intersection of two sets is the set of members which are in both of the given sets. In the two sets above, a is the only letter that is in both sets.

What will be the intersection of two sets if there is no member in one set that is also in the other? The intersection is the empty set, $\{\quad\}$.

Examples of intersections of sets:

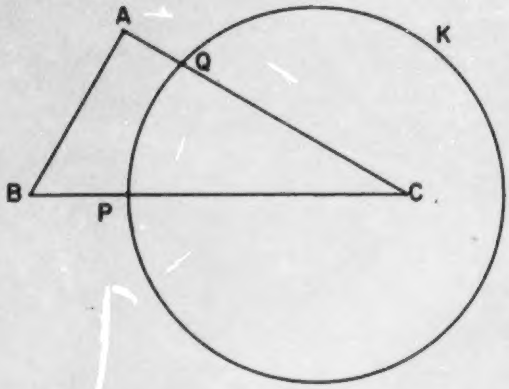
1. $S = \{2, 3, 4, 5\}$ $T = \{1, 2, 3\}$ $S \cap T = \{2, 3\}$
2. $A = \{a, b, c\}$ $B = \{c, d, e\}$ $A \cap B = \{c\}$
3. $M = \{a, e, i, o, u\}$ N = set of consonants in the alphabet

$$M \cap N = \{\quad\}.$$

4. In the drawing below are two geometric figures. One is a triangle and the other is a circle. There are two points, P and Q, which are on both the circle and on the triangle. So P and Q are the points of intersection.

We write this

$$(\triangle ABC) \cap (\text{Circle } K) = \{P, Q\}$$



5. In a certain school some of the pupils are studying English and some are studying History. There are some, but not all, of these pupils that are studying both English and History. Draw a Venn diagram to represent this situation.

BRAINTWISTER

6. In another school there are 47 pupils. The subjects they study are History, English, and Science.

4 students study all three subjects

5 students study History and English but not Science

6 students study History and Science but not English

7 students study English and Science but not History

7 students study History only

8 students study English only

10 students study Science only

Draw a Venn diagram to illustrate this situation. You will need to use 3 regions.

Exercise Set 2

1. If $S = \{a, b, 3, 5\}$, $T = \{\square, 3, c, d\}$, $W = \{5\}$, and $E = \{\ \}$, then find

a. $S \cap T =$ c. $T \cap E =$ e. $T \cap W =$

b. $S \cap W =$ d. $W \cap E =$ f. $S \cap E =$

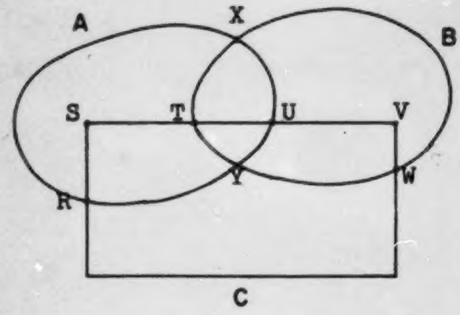
2. If $A = \{4, 3, 1, 2\}$, $C = \{3, 4, 2, 1\}$,
$B = \{a, 4, 7\}$, and $D = \{\square, 5\}$,

then find

a. $A \cap C =$ c. $C \cap D =$

b. $A \cap B =$ d. $A \cap F$, if $F = B \cap C$.

3. Trace a drawing similar to the one pictured below. The simple closed curves are labeled A, B, and C.

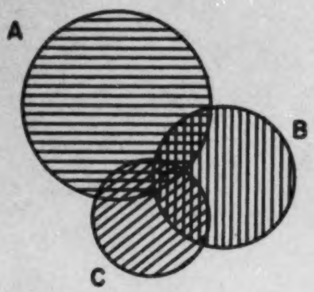


Find

a. $A \cap B =$ c. $B \cap C =$

b. $A \cap C =$ d. $(A \cap B) \cap C =$

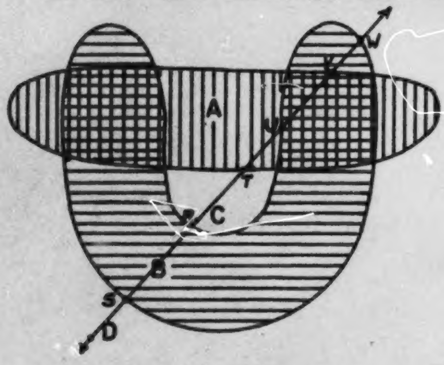
4. Draw circular regions A, B, C as shown. Draw horizontal segments in the region A as shown. Then draw vertical segments in the region B and slanting segments in the region C.



Describe how each of the following is lined.

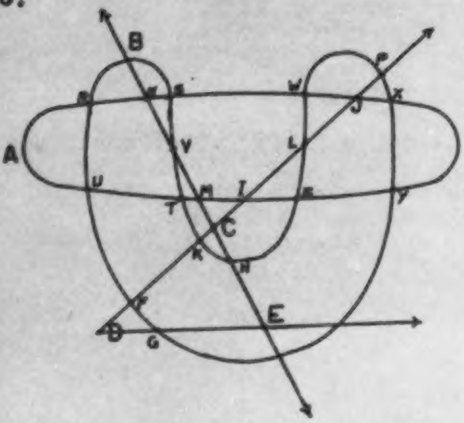
a. $A \cap B$ (lined with vertical and horizontal lines)
b. $B \cap C$
c. $A \cap C$
d. $A \cap (B \cap C)$

5. Look at the oval-shaped region A lined with vertical lines, the horseshoe-shaped region B lined with horizontal lines, and the line CD with some points labeled on it. Now describe each of the following intersection sets:



a. $A \cap B$
b. $A \cap \overrightarrow{CD}$
c. $A \cap \overrightarrow{DC}$
d. $B \cap \overrightarrow{CD}$
e. $B \cap \overleftrightarrow{CD}$
f. $A \cap \overline{CD}$

6.



Look at the oval-shaped curve A, the horseshoe-shaped curve B, the $\angle PDE$, the line BE and the points that are labeled.

Describe each of the following intersection sets.

a. $A \cap B$

b. $B \cap \overrightarrow{DC}$

c. $A \cap \overrightarrow{DC}$

d. $A \cap \overleftrightarrow{EC}$

e. $\overline{CD} \cap \angle DEC$

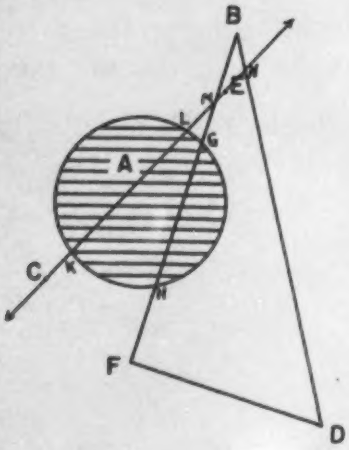
f. $A \cap \angle CDE$

g. $A \cap \{E\}$

h. $A \cap \{C, D, E\}$

i. $(A \cap B) \cap A$

7.



Look at the circular region A, the $\triangle$ BDF, the line CE, and the labeled points. Describe each of the following intersection sets.

a. $A \cap \triangle BDF$

b. $A \cap \overleftrightarrow{CE}$

c. $\triangle BDF \cap \overline{CE}$

d. $\triangle BDF \cap \overrightarrow{CE}$

e. $\triangle BDF \cap \overrightarrow{EC}$

f. $\triangle BDF \cap \{E, C\}$

UNION OF SETS

A class was making arrangements for a party. Two committees were set up. One was the committee for refreshments. Its members were Mary, Joan, and Bill. Call this set R.

We can write

$$R = \{\text{Mary, Joan, Bill}\}$$

The other committe was the committee for decorations. Its members were Harry, Bill, and Henry. Call this set D.

We can write

$$D = \{\text{Harry, Bill, Henry}\}$$

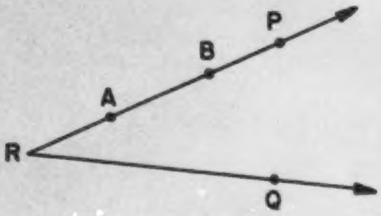
The pupils who are on committees are Mary, Joan, Bill, Harry, and Henry. This set is called the union of the two sets R and D. Its members are the members which are in either R or D or both.

We write this

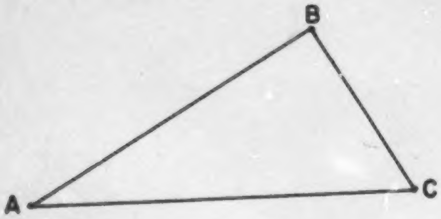
$$R \cup D = \{\text{Mary, Joan, Bill, Harry, Henry}\}$$

Each name appears only once in the set $R \cup D$. Bill is on both committees and each of the other pupils is on one of the committees.

Let us look at another example of the union of two sets. Look at the geometric figures in the next drawing. Think of the line segment AB and the angle PRQ. Is the set of points of $\overline{AB}$ contained in the set of points of $\angle$ PRQ?



What is the union of the set of points of $\overline{AB}$ and the set of points of the angle PRQ? Do you see that the points of $\overline{AB}$ are also points of the angle PRQ? The points of $\overline{AB}$ are points of both of the sets of points in the drawing. All the other points in the drawing are points of $\angle$ PRQ. Hence, the union of the two sets $\overline{AB}$ and $\angle$ PRQ is the $\angle$ PRQ.



Look at the triangle ABC in the figure shown above. It consists of the points of the three line segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$. We can say that the union of the three line segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ is the triangle ABC.

We write:

$$(\overline{AB} \cup \overline{BC}) \cup \overline{CA} = \triangle ABC.$$

Exercise Set 3

1. If $A = \{a, b, c, d\}$

   $B = \{c, 1, 2\ \triangle\ \}$

   $E = \{\ \}$

   Then what is

   a. $A \cup B$

   b. $A \cup E$

   c. $A \cup (A \cap B)$

   d. $B \cup (A \cap B)$

2. If A is the set of the five most commonly used vowels, and B is the set of the first 10 letters of the alphabet, and C is the set of the last 10 letters of the alphabet, then what is

   a. $A \cup B$

   b. $B \cup C$

   c. $A \cup \{w, y\}$

   d. $(A \cup B) \cap C$

   e. $(A \cap C) \cup (B \cap C)$

   f. $\{u, w, y\} \cup (A \cap B)$

3. If A is the set of students in your class, and B is the set of blue-eyed students in your class, and C is the set of red-haired students in your class, then what is described in each of the following:

   a. $A \cup B$

   b. $A \cup C$

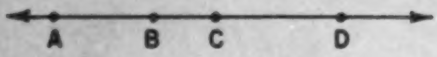
   c. $C \cup (A \cap B)$

   d. $A \cup (B \cup C)$

   e. $(A \cup B) \cup C$

   f. $A \cup (B \cap C)$

4.



Copy each statement a, b, c, d, and complete it so that it will be true.

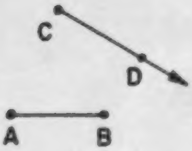
a. $\overline{AC} \cup \overline{BC} =$

b. $\overrightarrow{AB} \cup \overrightarrow{BC} =$

c. $\overrightarrow{AB} \cup \overrightarrow{BA} =$

d. $\overleftrightarrow{AC} \cup \overleftrightarrow{BC} =$

5. Copy each statement a, b, c, and complete it so that it will be true.



a. $\{A\} \cup \{C\} =$

b. $(\overline{CD} \cup \overline{BC}) \cup \overline{BD} =$

c. $(\overline{CD} \cup \overline{BC}) \cup (\overline{BD} \cup \overline{AB}) =$

6. Mark points A, B, and C (not in the same line) on your paper. Draw $(\overrightarrow{AB} \cup \overrightarrow{BC}) \cup \overrightarrow{CA}$. Is the figure a triangle?

Is a triangle a union of three rays?

7. Mark points D, E, and F (not in the same line) on your paper. Draw $(\angle DEF \cup \angle EFD) \cup \angle DEF$. Is the figure a triangle?

8. Mark points G, H, and I (not in the same line) on your paper. Draw $(\overline{GH} \cup \overline{HI}) \cup \overline{IG}$.

a. Is the figure a triangle?

b. Is a triangle a union of three line segments?

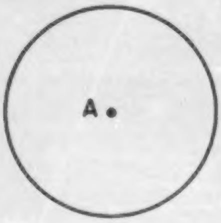
c. Is the union of three line segments always a triangle?

d. What is a triangle?

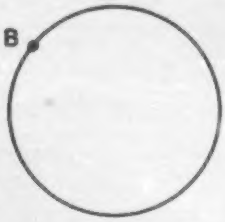
Naming Circles

You know how to draw a picture of a circle by using your compass. Or, if you are working at the chalkboard you will probably use a piece of string and a piece of chalk. The sharp point of the compass that is placed on your paper marks the center of the circle. All the points of the circle are the same distance from its center.

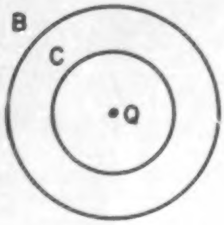
In the drawing below, the point A is the center of the circle. If you wish to speak of this circle you may call it Circle A.



Or, if you like you may mark some point of the circle as in this drawing and speak of the circle as Circle B.

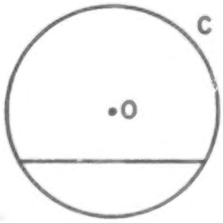


If you have two circles with the same center as in the next drawing, which will be the better way of speaking of them?



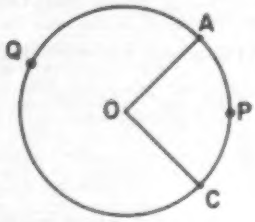
What is the intersection set of the two circles in the figure where the two circles have the same center Q? What is the union set of these two circles?

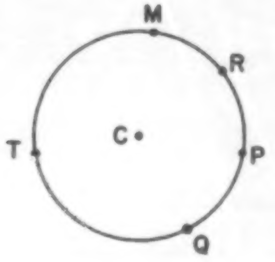
If you draw a line segment which has both of its endpoints on a circle, the line segment is called a chord of the circle. Draw a circle on your paper. Then draw a chord of the circle.



How many chords could be drawn in a circle? Is there more than one chord of a circle? Can you draw a chord of the circle which will pass through the point O? Can you draw more than one chord which will pass through the point O? The name for a chord which passes through O is diameter.

An arc of a circle is a particular subset of the circle. In the drawing below, the subset of the circle between A and C is called an arc. But there are two subsets of the circle between A and C. One subset is the arc on which we have marked the point P and the other is the subset on which we have marked the point Q. We can tell one from the other if we name one arc APC and the other arc AQC. We write these $\widehat{APC}$ and $\widehat{AQC}$. You see that we write the name of the arc in much the same way that we write the name of a line segment. But we need three letters to name an arc while two letters are enough to name a line segment.

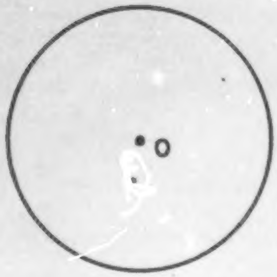




Write on your paper the names of some of the arcs of the cirrle whose center is C. Remember to use 3 letters in naming an arc.

Exercise Set 4

1.



a. Mark a point O on your paper and draw a circle as shown. Is O a point of the circle?

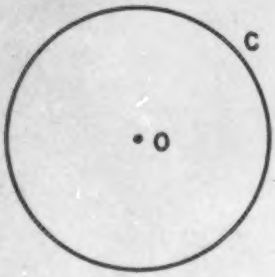
b. Mark points A, B, C, and D on the circle. Name at least three arcs of your circle.

c. Draw a chord of your circle. Draw another chord of your circle. Are the chords the same length?

d. Do you think this circle has a longest chord? If so, draw it.

e. Do you think each circle has a shortest chord? If so, draw it. If not, why not?

2.



a. Mark a point O on your paper and draw a circle as shown. Name your circle C. Is point O in the interior of C?

b. Mark a point A in the interior of your circle. Mark a point B which is not in the interior and is also not a point of the circle.

c. Is $\overline{AB}$ in the interior of your circle?

d. Is any point of $\overline{AB}$ a point of your circle?
Is $\overline{AB}$ a part of any circle?

e. $\overline{AB} \cap C =$?

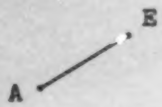
f. $\overleftrightarrow{AB} \cap C =$?

g. $\overrightarrow{AB} \cap C =$?

h. $\overrightarrow{BA} \cap C =$?

3. If ℓ represents a line and C represents a circle, then the set $\ell \cap C$ has how many members? Draw enough pictures to illustrate your answer.

4.



a. Draw $\overline{AE}$ on your paper.

b. Draw a circle with center at A and $\overline{AE}$ as a radius. Call the circle C.

c. Is a radius of a circle part of the circle? Why?

d. Draw a diameter of your circle. Is a diameter of a circle part of the circle?

5. a. Draw a circle with center marked A and a radius $\overline{AE}$.

b. Can you imagine another circle with center E and radius $\overline{AE}$? Is there more than one such circle?

6. Imagine two circles.

a. Do they have to be in the same plane?

b. Could their intersection be the empty set?

c. Could they intersect in exactly one point? Illustrate.

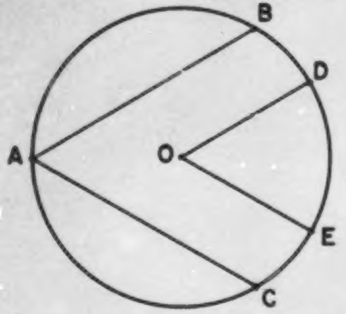
d. Could they intersect in exactly two points? Illustrate.

e. Could they intersect in exactly three points? Illustrate.

f. Could they intersect in more than three points? Illustrate.

INSCRIBED ANGLES AND CENTRAL ANGLES

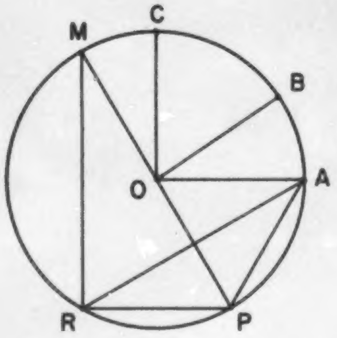
Draw two chords of a circle with point A as one endpoint of both. See the drawing below.



Call the chords AB and AC. The ∠ BAC is called an inscribed angle.

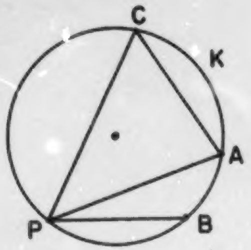
Angle DOE is called a central angle since its vertex is at the center of the circle.

Write the names of the central angles and the names of the inscribed angles that you see in the drawing below. A diameter of the circle is $\overline{MP}$. Make two columns on your paper as suggested below the drawing.



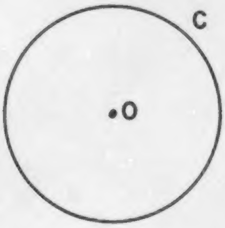
Central Angles	Inscribed Angles
Name at least 7 of these.	Name at least 8 of these.

If we have an inscribed angle in a circle, we can name the arc in its interior with two letters. In the table below the next drawing we have written the names of some of the inscribed angles and the arc which lies in the interior of the angle.



Inscribed Angle	Arc in its interior
∠ CPA	$\overset{\frown}{CA}$
∠ CPB	$\overset{\frown}{CB}$
∠ ACP	$\overset{\frown}{AP}$

Can you tell how to locate two points of the circle C in the drawing below so that the two arcs into which they divide the circle will be congruent?



After you think you have located two such points, ask your teacher if you have done it correctly.

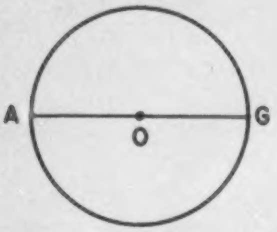
Exercise Set 5

1. a. Draw a circle with center marked O.

   b. Draw a central angle of the circle. Is this central angle a part of the circle?

   c. Draw an inscribed angle of the circle. Is it a part of the circle?

2. Draw a circle as shown below.



   a. Mark the center O and diameter $\overline{AG}$.

   b. Mark a point D on your circle.

   c. On $\overset{\frown}{ADG}$ mark points B, C, E, and F.

   d. Name at least 3 arcs of the circle.

   e. Mark and measure (with a protractor) $\angle$ ABG, $\angle$ ACG, $\angle$ $\overset{\frown}{ADG}$, $\angle$ AEG, and $\angle$ AFG.

   f. Do you notice anything surprising about the measures, in degrees, of these inscribed angles? Explain.

   g. Make a guess about the measures, in degrees, of all inscribed angles with one ray through one end of a diameter and the other ray through the other end.

3. a. Draw a circle with center marked O and diameter $\overline{AC}$.

   b. Mark a point B on the circle.

   c. Draw $\angle BOC$. What kind of angle is this?

   d. By using a protractor, approximate the measure of $\angle BOC$.

   e. Draw $\angle BAC$. What kind of angle is this?

   f. By using a protractor, approximate the measure of $\angle BAC$.

   g. Mark a point D of $\overset{\frown}{ACB}$ which is not a point of $\overset{\frown}{ABC}$. Draw inscribed $\angle BDC$.

   h. By using a protractor, approximate the measure of $\angle BDC$.

   i. Draw three more inscribed angles with vertices on $\overset{\frown}{ADC}$ each having one ray through B and the other one through C. Do you notice anything about the measures of these inscribed angles with common $\overset{\frown}{BC}$?

   j. Make a guess about measures of all inscribed angles of a given circle with the same $\overset{\frown}{BC}$.

   k. Make a guess about the measure of a central angle with $\overset{\frown}{BC}$ and the measure of an inscribed angle with $\overset{\frown}{BC}$.

4. Draw a picture of a circle on a piece of paper. Crumple this sheet into a wad. Is the drawing still a picture of a circle? Why?

5. **BRAINTWISTER**

Imagine all the diameters of a circle.

a. What would be their intersection?

b. What would be their union?

6. **BRAINTWISTER**

a. Place the sharp point of a compass at a point of one face of a block. Name this point A. With A as a center and a suitable setting on the compass, draw a simple closed curve on this face. Is the result a picture of a circle?

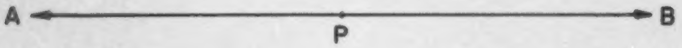
b. With this same setting of the compass, draw a simple closed curve on the box using the vertex B as the center. Are all the points of the resulting curve points of a circle? Why?

STRAIGHTEDGE AND COMPASS CONSTRUCTIONS

CONSTRUCTING RIGHT ANGLES USING PROTRACTOR

Let us use the protractor to make a right angle with its vertex at the point P on the line AB.

Make a drawing of $\overleftrightarrow{AB}$ on your paper and follow the directions suggested below



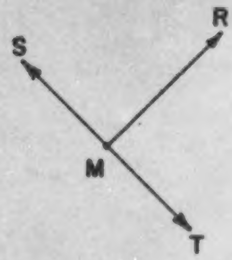
Place the V-point of the protractor at P and the zero ray of the protractor along $\overrightarrow{PB}$. Mark a point, call it Q, on the paper at the 90° mark on either scale of the protractor. Draw $\overrightarrow{PQ}$. Measure $\angle$ BPQ and $\angle$ APQ with the protractor. Is $m \angle BPQ = m \angle APQ = 90$? It should be.

In the drawing that you have made, you may say that $\overrightarrow{PQ}$ is perpendicular to $\overleftrightarrow{AB}$ at P. We write either $\overrightarrow{PQ}$ is $\perp$ to $\overleftrightarrow{AB}$ at P, or $\overleftrightarrow{AB}$ is $\perp$ to $\overrightarrow{PQ}$ at P. Notice the symbol $\perp$ means perpendicular. Saying $\overrightarrow{PQ}$ is $\perp$ to $\overleftrightarrow{AB}$ at P means that $\angle$s APQ and BPQ are right angles. Also saying $\angle$s QPB and QPA are right angles means $\overrightarrow{PQ}$ is $\perp$ to $\overleftrightarrow{AB}$ at P.

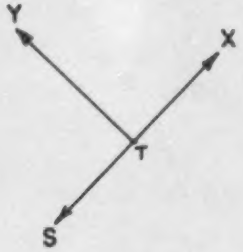
Exercise Set 6

1. Write statements for each drawing as written in a. The angles shown in the drawings are right angles.

a.

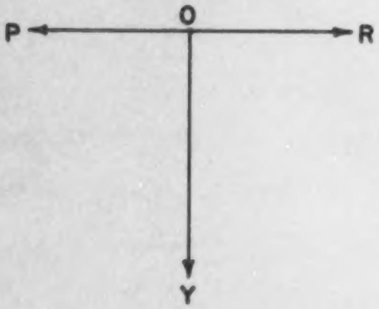


b.

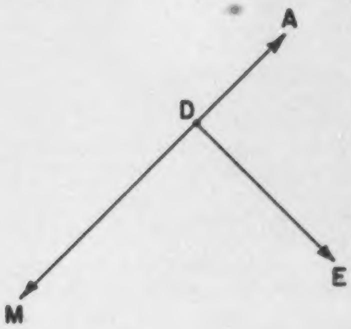


∠s SMR and ∠ RMT are right angles. $\overrightarrow{MR}$ is ⊥ to $\overleftrightarrow{ST}$ at M.

c.



d.

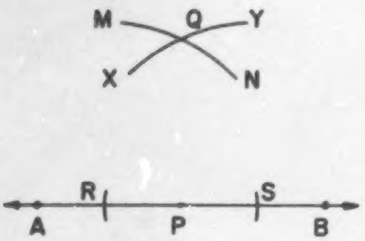


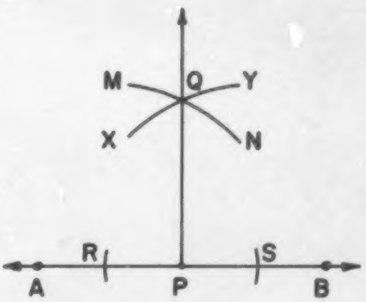
CONSTRUCTING A RAY PERPENDICULAR TO A LINE WITHOUT USING PROTRACTOR

Next let us see how we can draw a ray perpendicular to a line without using the protractor. We shall use a compass and a straightedge.

As you read the directions here, carry out the constructions on your paper. The marks in the drawing will help you to follow instructions.

Choose any point, P, on line AB.





Set the compass point at P and draw two small arcs which will cut $\overleftrightarrow{AB}$ at points R and S. Is $PR \cong PS$?

Set the compass point at R. Adjust the compass so that the compass setting is greater than the length of $\overline{RP}$. Then draw an arc such as the arc MN in the drawing.

Keep the same compass setting. Set the compass point at S and draw another arc XY in the drawing. The arcs intersect in a point. Call it Q.

Draw $\overrightarrow{PQ}$. The ray $\overrightarrow{PQ}$ is perpendicular to $\overleftrightarrow{AB}$ at P. Your completed drawing should look something like the one above at the right.

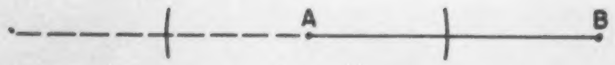
Do you think $\angle SPQ$ and $\angle RPQ$ are right angles? Measure each of them with the protractor. The measure in degrees of each angle should be 90.

This is equivalent to saying that $\overrightarrow{PQ}$ is $\perp$ to $\overleftrightarrow{AB}$ at P from the meaning of perpendicular.

Ray PQ is $\perp$ to $\overleftrightarrow{AB}$. The $\overline{PQ}$ is part of $\overrightarrow{PQ}$, and $\overline{RS}$ is part of $\overleftrightarrow{AB}$. So we can say $\overline{PQ}$ is $\perp$ to $\overline{RS}$ at point P.

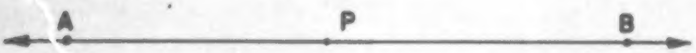
Suppose you wish to construct a line segment $\perp$ to $\overline{AB}$ at a point on $\overline{AB}$. Can you make the construction in the same way that you made the $\overrightarrow{PQ}$ $\perp$ to $\overleftrightarrow{AB}$?

Next, suppose you wish to construct a line segment $\perp$ to $\overline{AB}$ at point A. Does the drawing below show you how to begin? The dotted segment to the left of A suggests that it was not a part of $\overline{AB}$ but that you needed it in order to make the construction. Now complete the construction on your paper.

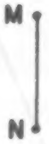


Exercise Set 7

1. Copy $\overleftrightarrow{AB}$ on your paper and construct a ray $\perp$ to $\overleftrightarrow{AB}$ at P. Make your construction so that the ray will be in the half plane below $\overleftrightarrow{AB}$.

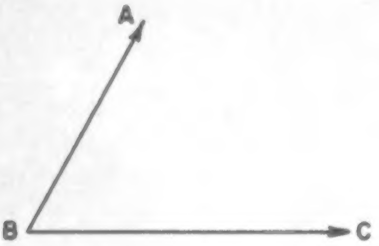


2. Copy the segment MN on your paper and construct a segment $\perp$ to $\overline{MN}$ at point N. Make your construction so that the segment will be to the right of the point N.



BISECTING AN ANGLE

The $\angle ABC$ in the drawing below has the measure in degrees of 60. This angle can be made by using the protractor or by making an equilateral triangle. Remember we know the measure in degrees of any one of the angles of any equilateral triangle is 60.



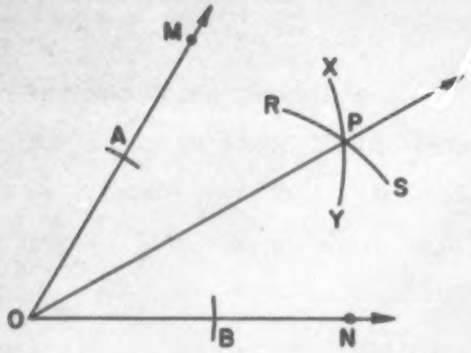
Copy the angle on your paper.

Draw $\overrightarrow{BK}$ so that the measure in degrees of $\angle CBK$ is 30. Then the measure in degrees of $\angle ABK$ is also 30.

We say that $\overrightarrow{BK}$ bisects $\angle ABC$. The measures of $\angle CBK$ and $\angle ABK$ are the same. The sum of their measures in degrees is the measure in degrees of $\angle ABC$.

If you have an angle drawn on your paper, could you always use the protractor to help you find a ray which bisects the angle? You could do this if you read the scale on the protractor very accurately.

Now we wish to show how to draw a ray without using the protractor so that the ray will bisect the angle. As you read the directions on the following page, carry out the constructions on your paper. The marks in the next drawing will help you to follow the instructions.



The angle that we are going to bisect is $\angle$ MON. Here are the steps in the construction of the ray that is the bisector.

With the point of the compass set on point O, draw two arcs so that one of them cuts $\overrightarrow{OM}$ at a point (call it A) and the other arc cuts $\overrightarrow{ON}$ at a point (call it B). Keep the setting of the compass unchanged while drawing the two arcs.

With the point of the compass set on point A, and the setting of the compass so that it is greater than one-half the distance from A to B, draw an arc such as $\overset{\frown}{XY}$.

With the point of the compass at B, and the setting of the compass the same as used in drawing the arc XY, draw another arc such as $\overset{\frown}{RS}$.

The two arcs you have just drawn intersect at a point P. Draw $\overrightarrow{OP}$.

The $\overrightarrow{OP}$ bisects the $\angle$ MON.

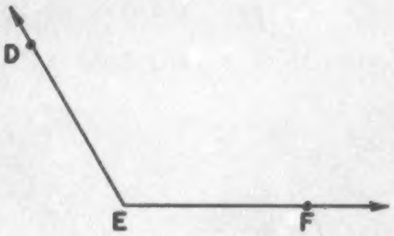
Measure $\angle$ POB and $\angle$ POA with your protractor.

Is m$\angle$ POB in degrees equal m$\angle$ AOP in degrees?

Is m$\angle$ POB + m$\angle$ AOP = m$\angle$ AOB?

Exercise Set 8

1. Copy the angle DEF on your paper. Then construct $\overrightarrow{EH}$ so that $\overrightarrow{EH}$ will bisect the angle. Use your compass and straightedge but not your protractor.



2. With your protractor measure ∠ DEF in the drawing on your paper. Then ∠ HEF and ∠ HED. Is the sum of the measures in degrees of ∠ HEF and ∠ HED equal to the measure of ∠ DEF?

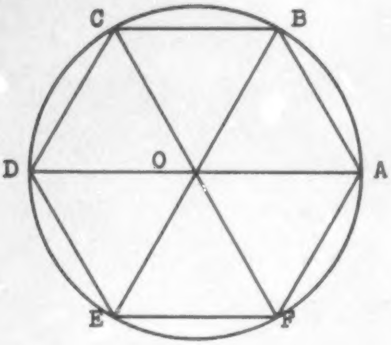
3. On your paper draw $\overline{AB}$ about 4 inches in length. Name a point O on $\overline{AB}$ between A and B.
Construct $\overline{OR}$ $\perp$ to $\overline{OA}$ at O. Use the straightedge and compass but not the protractor.

4. In the drawing you made for Exercise 3, construct a ray OK so it will bisect ∠ AOR. Use the straightedge and compass but not the protractor.

5. Find by using the protractor the measure of ∠ AOR that you constructed in Exercise 3. Then use the protractor to find the measure of ∠ AOK. Is the measure of ∠ AOR twice as large as the measure of ∠ AOK?

INSCRIBING A REGULAR HEXAGON IN A CIRCLE

Look very carefully at the drawing below. We want you to see some of the many interesting things about the drawing.



Things to be seen in the drawing:

1. A circle with center at point O.
2. Segments OA, OB, OC, OD, OE, and OF are radii of the circle.
3. There are 6 triangles shown with O as a common vertex.
4. Two sides of each of the 6 triangles are radii.
5. One side of each of the 6 triangles is a chord of the circle.
6. The vertices of the polygon ABCDEF are points of the circle.
7. The chords AB, BC, CD, DE, EF, and FA seem to be congruent.
8. All of these 6 chords seem to be of the same length as the radii.
9. The 6 triangles seem to be equilateral triangles.
10. The 6 triangles seem to be congruent to each other.
11. The segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ seem to be diameters.
12. The measure in degrees of each of the angles ABC, BCD, CDE, DEF, EFA, FAB seems to be 120.

After you have looked carefully, did you see all of the things that are listed following the drawing? If you saw all twelve of the things listed above and below the drawing, you did very well. But you may have seen more than these. If you did, tell your teacher. Your teacher may have seen some more, too.

As you looked at the drawing, you may have discovered how the drawing can be constructed with the use of compass and straightedge.

The following directions describe how the drawing can be made with compass and straightedge. Follow them carefully and make the construction on your paper. Then save it so that you can use it later.

First draw the circle with center O. Make its radius the length of $\overline{OA}$. Mark a point of your circle and call it A.

Put compass point at A. Use the same compass setting as used in drawing the circle. Make an arc that cuts the circle at point B.

Next put compass point at B, keeping the same compass setting. Make an arc that cuts the circle at C.

Now you see how to mark the point that is named D. Mark it.

Before you mark any of the other points of the circle, do the following:

Draw $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$, $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$.

Are the triangles OAB, ABC, and OCD equilateral triangles? (Yes.) You know that they are because the three sides of each triangle are congruent to each other. Also, the length of each side of the 3 triangles is the same as the length of the radius of the circle.

Measure each angle of one of the triangles with your protractor. The measure, in degrees, of each angle should be 60.

Is $m \angle AOB + m \angle BOC + m \angle COD = 180$? (It should be.)

Place your straightedge along the $\overline{OA}$. Are $\overline{OA}$ and $\overline{OD}$ parts of the same straight line? They should be. Segment AOD is a diameter of the circle. In this unit we will read a line segment that is a diameter with three letters. The letter in the middle names the point at the center of the circle.

Now continue from D to mark points E and F just as you marked points B and C when you started from A.

Draw $\overline{OE}$, $\overline{OF}$, $\overline{DE}$, $\overline{EF}$.

Measure $\angle DOE$ and $\angle EOF$ with your protractor. Is the $m \angle DOE + m \angle EOF = 120$? It should be.

Is the $m \angle FOA = 60$? It should be. With the compass setting the same as the length of the radius of the circle and the point of the compass at F, make an arc. This arc should intersect the circle at A.

One more equilateral triangle each of whose sides is congruent to a radius of the circle can be "fitted into" the circle. One vertex will be at O, one at F, and the third one at A. This completes the construction.

Now you can see that all the "seem to be" statements are true. These statements are numbered 7, 8, 9, 10, 11, 12.

The polygon ABCDEF is a hexagon. It is called a hexagon because it has six sides. In this hexagon the sides are congruent to each other, and the angles are congruent to each other.

A polygon which has congruent sides and congruent angles is a regular polygon. The polygon you have drawn is a regular hexagon.

Can we say that the polygon ABCDEF is a regular hexagon inscribed in the circle that has its center at O and that the length of each side of the hexagon is the same as the length of the radius of the circle?

Exercise Set 9

(You will need to use the drawing of the inscribed hexagon that you have constructed or the one drawn in this book.)

1. If the length of the radius of the circle is 1 inch, what is the measure in inches of the perimeter of the hexagon?

2. What is the measure in degrees of each of the angles AOB, BOC, COD, DOE, EOF, and FOA?

3. Draw $\overline{AC}$, $\overline{CE}$, and $\overline{EA}$. Use your compass to compare the lengths of these segments. Are $\overline{AC}$, $\overline{CE}$, and $\overline{EA}$ congruent? What kind of triangle do we call $\triangle$ ACE?

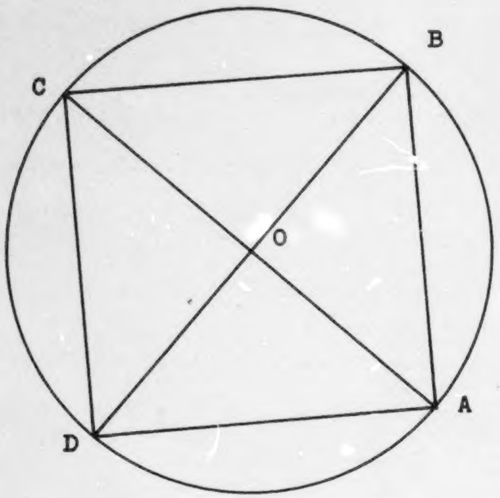
4. Construct a ray (call it $\overrightarrow{OP}$) which bisects $\angle$ AOB. Mark the point Q in which $\overrightarrow{OP}$ intersects the circle. Draw $\overline{AQ}$ and $\overline{BQ}$. Use your compass to compare their lengths. Is $\overline{AQ} \cong \overline{BQ}$?

5. Does exercise 4 suggest a way that you could use to construct a polygon with 12 congruent sides that would be inscribed in the circle? Describe the way you would do it. Do not make the construction.

6. Does exercise 5 suggest a way that you could use to construc a polygon with 24 congruent sides that would be inscribed in the circle? Describe the way you would do it. Do not make the construction.

INSCRIBING A SQUARE IN A CIRCLE

Look carefully at the drawing but do not do any measuring now. Make a list of some of the things you see. After you have studied the drawing, then see whether you saw all the things that are listed below the drawing.



Things to be seen in the drawing:

1. A circle with its center at point O.
2. $\overline{AOC}$ and $\overline{BOD}$ are diameters of the circle.
3. $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$ are radii of the circle.
4. $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are chords of the circle.
5. Four triangles OAB, OBC, OCD, and ODA with a common vertex at O.
6. Four triangles ABC, BCD, CDA, and DAB.
7. The angles AOB, BOC, COD, and DOA seem to be right angles.
8. The angles ABC, BCD, CDA, and DAB seem to be right angles.
9. The chords $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ seem to be congruent.

As you looked at the drawing, you may have discovered how it can be constructed with compass and straightedge.

The following directions describe how the construction can be carried out. Follow them carefully and make the construction on your paper. Then save it so you can use it later.

Draw a circle on your paper which will have its radius the same length as the circle in the drawing. Call the center of your circle O and mark a point A on your circle that corresponds to point A in the drawing.

Draw the diameter that has one of its ends at A. Name the other end of the diameter C. Must the center O be a point of the diameter?

Construct another diameter which will be $\perp$ to $\overline{AC}$ at O. Use your compass and straightedge but not your protractor. (Remember that you know how to construct a line $\perp$ to another line at a point on it.) Letter the ends of this diameter as in the drawing.

Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. This completes the construction.

Exercise Set 10

1. Use your compass to compare $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. Are they congruent? They should be.

2. Use your protractor to measure $\angle$ AOB, $\angle$ BOC, $\angle$ COD, and $\angle$ DOA. Is 90 the measure in degrees of each of them? Is each one of them a right angle?

3. Use your protractor to measure the $\angle$ ABC, $\angle$ BCD, $\angle$ CDA, and $\angle$ DAB.
   Is 90 the measure in degrees of each one of them?
   Is each one of them a right angle?
   Is each one of them inscribed in a semi-circle? (Remember that an angle is inscribed in a semi-circle if its vertex is a point of the circle and the endpoints of a diameter are points of its rays.)

4. Name eight right triangles in the drawing.

5. Name eight isosceles triangles in the drawing.

6. Are the triangles you named in exercise 4 the same as you named in Exercise 5?

7. Do you see any equilateral triangles in the drawing?

8. Can we say that this quadrilateral ABCD is a regular inscribed polygon? Is it a square?

9. With compass and straightedge construct the bisector of $\angle$ AOB. Let P be the name of the point where the bisector intersects the circle.

10. Draw the ray opposite to $\overrightarrow{OP}$ until it intersects the circle at a point between C and D. Name the point Q.

11. Is 45 the measure in degrees of each of the angles AOP, BOP, COQ, and DOQ?

12. Draw $\overline{AP}$, $\overline{BP}$, $\overline{CQ}$, and $\overline{DQ}$. Use your compass to compare their lengths. Are they congruent segments?

13. Put the compass point at B. Use the same setting of your compass that you used in Exercise 12 when you compared the length of the 4 segments. Make an arc which will intersect the circle between B and C. Name this poing R. Draw $\overline{BR}$ and $\overline{CR}$.

14. Draw the diameter which has one end at point R. Let S be the name of the point at its other end. Draw $\overline{DS}$ and and $\overline{SA}$.

15. Is the polygon APBRCQDS an inscribed polygon? It has eight sides. It is called an octagon. All of its sides are congruent line segments.

16. Use your protractor to measure at least three angles of the octagon. For example, you might measure angles APB, PBR, and BRC. Is 135 the measure in degrees of each one of them?

17. Can you tell how you could inscribe a polygon of 16 sides in the circle so that all of its sides would be congruent segments? Describe how it can be done but do not make the drawing.

18. You have seen how you could inscribe polygons of 4, 8, and 16 sides in the circle. Can you describe how you could inscribe a polygon of more than 16 sides in the circle so that all of the sides of the polygon would be congruent line segments?

19. Do you think the perimeter of the octagon is greater than the perimeter of the square? of the 16-sided polygon?

20. Do you think we could inscribe in a circle a polygon with a very, very large number of congruent sides so that its perimeter would be nearly the same as the perimeter of the circle? Exactly the same?

Chapter 10

REVIEW

WORKING WITH EXPONENTS

Exercise Set 1

Write the numerals 1 through 12 on your paper. Decide if each of the following mathematical sentences is true or false. Then write true or false opposite each numeral on your paper.

1. $3^3 = 3 \times 3$
2. $4^5 = 4 \times 4 \times 4 \times 4 \times 4$
3. $3 \times 2^2 = 3 \times 2 \times 3 \times 2$
4. $5^3 \times 5^2 = 5^6$
5. $8 \times 2^3 = 8 \times 2 \times 2 \times 2$
6. $(2 + 3)^4 = 2^4 + 3^4$
7. $2^8 = 8^2$
8. $2^4 = 4^2$
9. $6^2 = 2^6$
10. $8^2 - 3^2 = (8 - 3)^2$
11. $8 \times 3^2 = (8 \times 3)^2$
12. $4 \times 3^2 = 4 \times 4 \times 3 \times 3$

16 : 71

$\frac{2}{3}$ $\frac{2}{2} \times \frac{2}{3} \times \frac{2}{2} \times \frac{3}{3}$ $\frac{16}{71}$

In Exercise 13 through 24 write each expression as a decimal numeral.

13. $3^2 + 1$
14. $11 - 3^2$
15. $2^4 - 4^2$
16. $8^2 - 6^2$
17. $8^3 - 6^3$
18. $(3^4 + 1) - 7^2$
19. $5^3 - (4^2 - 3^2)$
20. $(3 + 4)^2 + 5$
21. $88 - (3 - 3)^8$
22. $(18 - 17)^{10} + (2 + 1)^4$
23. $2^2 + (4^2 + 2^2)$
24. $4^2 - (3^2 - 2^2)^4$

Write >, or <, or = for each blank in Exercise 25 through 38 so that each mathematical sentence will be true.

25. $3^2 + 1$ ____ $1 + 3^2$

26. $18^5 + 21^7$ ____ $21^7 + 18^5$

27. $2^{10} + (2^{15} + 2^{21})$ ____ $(2^{10} + 2^{15}) + 2^{21}$

28. $9^{25} + 0$ ____ $0 + 9^{25}$

29. 64^{20} ____ $64^{20} \times 1$

30. $10^8 - 1$ ____ $10^8 + 1$

31. $15^7 + 8^7$ ____ $15^7 - 8^7$

32. $(4 \times 5)^2$ ____ $4^2 \times 5^2$

33. $6^2 - 4^2$ ____ $(6 - 4) \times (6 + 4)$

34. 0^{896} ____ 0^{895}

35. $7^2 + 5^2$ ____ $(7 + 5)^2$

36. 1^{502} ____ 502^1

37. $(1 + 1 + 1)^4$ ____ 3^4

38. $(1 - 1 + 1)^8$ ____ 0

Exercise Set 2

1. What counting number does n stand for if $4^n = 16$?

You could think, "$4^1 = 4$, so n is not 1.

$4^2 = 16$, so $n = 2$."

What counting number does n stand for so that each of these mathematical sentences will be true. Write your work as shown for Exercise a.

a. $\underline{4^n = 16}$ $\underline{4 \times 4 = 16}$ $\underline{4^2 = 16}$ $\underline{n = 2}$

b. $7^n = 49$

c. $2^n = 8$

d. $2^n = 32$

e. $10^n = 1000$

f. $6^n = 216$

g. $12^n = 144$

h. $4^n = 1024$

i. $3^n = 243$

2. Write each of these expressions as a decimal numeral.

a. n^2, if $n = 6$

b. 2^n, if $n = 6$

c. n^3, if $n = 5$

d. 3^n, if $n = 5$

e. $3^n + 4$, if $n = 2$

f. $6^n + n^6$, if $n = 2$

g. $3^n - 3^n$, if $n = 4$

h. $100^n - n^{100}$, if $n = 1$

3. What counting number does n stand for so that $2^n + 1 = 9$? You could think $2^1 + 1 = 3$, so n is not 1. $2^2 + 1 = 5$, so n is not 2. $2^3 + 1 = 9$, so $n = 3$.

What counting number does n stand for so that each of the mathematical sentences Exercise a through i will be true? Write your work as shown for Exercise a.

a. $3^n - 1 = 8$ $\quad (3 \times 3) - 1 = 8$ $\quad 3^2 - 1 = 8$ $\quad n = 2$

b. $2^n - 1 = 3$

c. $4^n + 3 = 19$

d. $6^n - 4 = 2$

e. $2^n + 1 = 33$

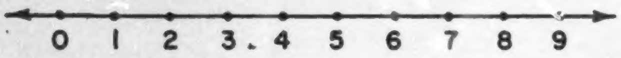
f. $5^n - 2^3 = 17$

g. $2^n - 2^4 = 16$

h. $2^n - 2^5 = 0$

i. $3^n + 10 = 37$

Exercise Set 3



Above is a graph of a set of whole numbers. Let us call it Set A. Set $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. In Exercise 1 and 2 you will be thinking about these whole numbers and only these whole numbers.

Think about the whole numbers in Set A.

Which are described by the mathematical sentence

$$n + 2 = 6?$$

The answer is 4, since $4 + 2 = 6$.

Which are described by $n < 2$?

The answer is 1 and 0, since

$$1 < 2 \quad \text{and} \quad 0 < 2.$$

1. Which members of Set A are described in Exercise a through j.

a. $n > 2$

b. $n - 2 = 5$

c. $n < 0$

d. $n > (5 + 2)$

e. $2 + n < 4$

f. $n > 5$ and < 7

g. $n < 2$ and > 0

h. $n < 6$ and > 7

i. $n > 3$ and < 7

j. $n < 6$ and > 7 (be careful)

The mathematical sentence $6 < 8 < 11$ means 6 is less than 8 and 8 is less than 11. $6 < 8 < 11$ can be written $6 < 8$ and $8 < 11$.

The mathematical sentence $3 < n < 5$ means to find n so n is greater than 3 and n is less than 5. The only member of Set A for which this is true is 4.

2. Which members of Set A are described in Exercise a through i?

a. $2 < n < 4$

b. $0 < n < 2$

c. $5 < n < 7$

d. $2 < n < 6$

e. $5 < n < 8$

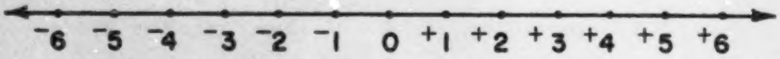
f. $3 < n < 7$

g. $0 < n < 1$

h. $6 < n < 7$

i. $7 < n < 8$

Exercise Set 4



Above is a number line of a set of integers. The members of the set are the integers that are greater than $^-7$ and less than $^+7$. Call this Set I.

1. Which of these Sets are shown on the above number line?

   a. All integers that are greater than $^-1$ and less than $^+4$.

   b. All integers that are less than $^-2$ and greater than $^-5$.

   c. All integers that are greater than $^-4$ and greater than $^+4$.

   d. All integers that are less than 0.

   e. All integers that are greater than $^-5$ and less than $^-3$.

2. What integers, if any, of Set I are described by these statements?

   a. All integers less than $^{+}2$ and greater than 0.

   b. All integers less than $^{-}5$ and greater than $^{-}4$.

   c. All integers greater than 0 and less than 0.

   d. All integers less than $^{+}6$ and greater than $^{+}3$.

3. If $^{-}2 < n < {}^{+}1$, and n is an integer, which integers are greater than $^{-}2$ and less than $^{+}1$? The answer is $^{-}1$ and 0.

4. For each of these, find n if n is a member of the set of integers.

   $\{^{-}6, {}^{-}5, {}^{-}4, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, {}^{+}3, {}^{+}4, {}^{+}5, {}^{+}6\}$

   a. $^{+}4 > n > {}^{+}2$

   b. $^{-}3 < n < {}^{+}2$

   c. $^{+}3 < n < {}^{+}6$

   d. $^{-}5 < n < {}^{-}2$

   e. $^{-}3 < n < 0$

   f. $^{-}1 < n < {}^{+}3$

Exercise Set 5

Answer each of these questions. You may write a mathematical sentence as you answer if you wish to do so.

1. What integer is 2 greater than the opposite of $^+4$?

2. What integer is 2 less than the opposite of $^-4$?

3. The integer $^+2$ is 2 greater than a certain integer which we will represent by A. Also $^+2$ is 2 greater than the opposite of A. What integer does A represent?

4. What integer is 6 greater than the opposite of $^+2$?

5. The sum of two integers is $^+5$. Both addends are positive integers. Write all possible mathematical sentences.

6. The sum of two integers is $^-6$. Both addends are negative integers. Write all possible mathematical sentences.

7. The sum of two integers is $^+8$. Both addends are negative integers. What are the addends? Does this problem have an answer? If your answer is no, state a reason.

Exercise Set 6

On a sheet of graph paper, draw the x-axis and the y-axis. Locate the points described below and label each point with its name, but not with its ordered pair.

1. Point E is 5 units above the x-axis and 4 units to the right of the y-axis.

2. Point R is on the y-axis and 8 units below the point whose coordinates are (0, $^{+}13$).

3. The x-coordinate of C is the opposite of $^{+}8$. Its y-coordinate is the opposite of $^{-}5$.

4. Point T is the reflection of the point with coordinates ($^{-}12$, $^{+}5$) in the y-axis.

5. Another point, R, is 6 units above and 9 units to the left of the point with coordinates ($^{+}7$, $^{-}1$)

6. Another point, C, is on the line segment joining E and T. It is midway between E and T.

7. Point O has as coordinates integers that are opposites. The second member of this ordered pair is the opposite of $^{-}5$.

If you have located the points correctly you will find that:

a. All points are on the same straight line.

b. A word is spelled by the letters. Write the word.

Exercise Set 7

The Ancient Egyptians used only unit fractions, for the most part. A unit fraction is a fraction which names a rational number such as $\frac{1}{2}$, $\frac{1}{9}$, $\frac{1}{15}$, or $\frac{1}{297}$ that has a numerator of 1.

The Egyptians did not have a symbol for rational numbers such as $\frac{3}{4}$. They had to think of $\frac{3}{4}$ as $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$, the sum of rational numbers named by unit fractions. They could have thought of $\frac{3}{4}$ as $\frac{1}{2} + \frac{1}{4}$.

In the exercises below you will be asked to name some numbers as the sum of two or more rational numbers whose numerators are 1. For this Exercise Set you are not to use $\frac{1}{1}$ as a unit fraction. You may have to try a few times for each exercise before you get the right answer.

1. Write each of these as the sum of two rational numbers named by unit fractions.

   a. $\frac{2}{5}$ b. $\frac{2}{8}$ c. $\frac{1}{2}$ d. $\frac{1}{3}$ e. $\frac{10}{20}$

2. Write each of these as the sum of two rational numbers named by fractions in two different ways. An answer for $\frac{3}{8}$ is shown in the box.

   a. $\frac{2}{3}$ b. $\frac{5}{12}$ c. $\frac{7}{12}$

$$\frac{3}{8} = \frac{1}{4} + \frac{1}{8}$$

$$\frac{3}{8} = \frac{1}{3} + \frac{1}{24}$$

3. Copy and fill in the blanks with rational numbers named by unit fractions.

$$\frac{3}{8} = \frac{1}{4} + \frac{1}{8}$$

a. $\frac{3}{8} = \frac{1}{4} + \frac{1}{16} + \underline{\qquad}$

b. $\frac{3}{8} = \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \underline{\qquad}$

c. $\frac{3}{8} = \underline{\qquad} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}$

4. Write $\frac{2}{3}$ as the sum of two equal rational numbers named by unit fractions.

Write $\frac{2}{3}$ as the sum of four equal rational numbers named by unit fractions.

5. Write $\frac{3}{4}$ as the sum of two rational numbers named by unit fractions; as the sum of three rational numbers named by unit fractions; as the sum of four rational numbers named by unit fractions.

BRAINTWISTERS

6. What is the largest number you can write as the sum of two rational numbers named by unit fractions?

7. What is the largest number you can write as the sum of two rational numbers named by two different unit fractions?

8. What is the largest number you can write as the sum of three rational numbers named by three different unit fractions?

Exercise Set 8

Copy the product expressions shown below.

a. Draw a line under each for which the product is less than the first factor.

b. Draw a ring around each for which the product is greater than the second factor.

	(a)	(b)	(c)
1.	$\frac{2}{3} \times 1\frac{7}{8}$	$3\frac{2}{5} \times \frac{1}{2}$	$\frac{3}{4} \times \frac{1}{8}$
2.	$12\frac{9}{11} \times \frac{4}{5}$	$\frac{1}{8} \times \frac{1}{9}$	$\frac{4}{9} \times \frac{9}{4}$
3.	$5\frac{2}{6} \times \frac{11}{4}$	$1\frac{7}{8} \times \frac{9}{8}$	$\frac{15}{4} \times \frac{3}{8}$
4.	$\frac{2}{3} \times \frac{9}{2}$	$\frac{11}{3} \times \frac{3}{11}$	$\frac{15}{8} \times \frac{1}{3}$

5. Study your answers for the above exercises. Then copy and complete these statements with the words "greater" or "less" in the blanks. Answers are about products of two factors.

a. If the first factor is less than 1, the product is _____ than the second factor.

b. If the second factor is less than 1, the product is ______ than the first factor.

c. If the first factor is greater than 1, the product is ______ than the second factor.

d. If the second factor is greater than 1, the product is ______ than the first factor.

Exercise Set 9

1. What number is n so each mathematical sentence is true?

a. $\frac{1}{2} \times \frac{2}{3} = n$ b. $\frac{7}{4} \times \frac{3}{8} = n$ c. $\frac{5}{6} \times \frac{9}{2} = n$

d. $1\frac{1}{2} \times \frac{5}{3} = n$ e. $3\frac{2}{5} \times 1\frac{3}{10} = n$ f. $2\frac{3}{4} \times 5\frac{1}{3} = n$

g. $\frac{1}{2} \times \frac{3}{2} = n$ h. $\frac{3}{5} \times \frac{7}{4} = n$ i. $\frac{1}{6} \times 2\frac{1}{2} = n$

j. $3\frac{3}{4} \times \frac{5}{2} = n$ k. $\frac{9}{7} \times 1\frac{1}{3} = n$ l. $\frac{11}{4} \times 2\frac{1}{5} = n$

2. Find a fraction name for each of the following.

a. $(4 \times n) + 1$ if $n = \frac{1}{2}$ e. $\left(\frac{5}{4} \times n\right) + \frac{7}{3}$ if $n = \frac{4}{3}$

b. $\left(n \times \frac{2}{3}\right) - 1$ if $n = \frac{3}{2}$ f. $\left(n \times \frac{7}{10}\right) + \frac{8}{20}$ if $n = \frac{1}{2}$

c. $\left(\frac{1}{2} \times n\right) + \frac{1}{2}$ if $n = \frac{1}{3}$ g. $\frac{13}{3} - (n \times 2)$ if $n = \frac{5}{3}$

d. $\frac{6}{3} - (n \times 2)$ if $n = \frac{2}{3}$ h. $n - \left(\frac{4}{5} \times \frac{2}{3}\right)$ if $n = \frac{14}{15}$

3. Whole-number exponents may be used with fractions as the base. For example,

the meaning of $\left(\frac{3}{4}\right)^2$ and $\left(\frac{2}{5}\right)^3$ is:

$\left(\frac{3}{4}\right)^2 = \frac{3}{4} \times \frac{3}{4}$ $\left(\frac{2}{5}\right)^3 = \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$

Write a fraction name for each of the expressions in Exercise a through l.

a. $\left(\frac{1}{2}\right)^2$

b. $\left(\frac{5}{6}\right)^2$

c. $\left(\frac{2}{3}\right)^3$

d. $\left(\frac{7}{8}\right)^2$

e. $\left(\frac{7}{2}\right)^2$

f. $\left(\frac{8}{3}\right)^2 - 1$

g. $\left(\frac{5}{2}\right)^3 + 1$

h. $1 - \left(\frac{1}{2}\right)^4$

i. $\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2$

j. $\left(\frac{4}{5}\right)^2 + \left(\frac{2}{5}\right)^2$

k. $\left(\frac{9}{10}\right)^2 + \left(\frac{11}{10}\right)^2$

l. $\frac{1}{2} + \left(\frac{3}{2}\right)^2$

Exercise Set 10

The mathematical sentence $\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$ is a statement about a relation among $\frac{2}{3}$, $\frac{5}{7}$ and $\frac{10}{21}$. The mathematical sentences

$$\frac{10}{21} \div \frac{5}{7} = \frac{2}{3}$$

or

$$\frac{10}{21} \div \frac{2}{3} = \frac{5}{7}$$

are also statements about a relation among the same rational numbers $\frac{2}{3}$, $\frac{5}{7}$ and $\frac{10}{21}$.

Similarly $\frac{1}{2} \times n = \frac{3}{5}$ and $n = \frac{3}{5} \div \frac{1}{2}$ are both statements about a relation among $\frac{3}{5}$, the product, $\frac{1}{2}$, the known factor and n, the unknown factor. One way of finding the unknown factor, n, so $\frac{1}{2} \times n = \frac{3}{5}$ is shown in the box below.

$$\boxed{\begin{aligned} \tfrac{1}{2} \times n &= \tfrac{3}{5} \\ n &= \tfrac{3}{5} \div \tfrac{1}{2} \\ &= \tfrac{3}{5} \times \tfrac{2}{1} = \tfrac{6}{5} \end{aligned}}$$

1. In Exercise a through f find n so that each mathematical sentence is true.

a. $\frac{3}{4} \times n = 6$

b. $n \times \frac{1}{5} = 8$

c. $\frac{2}{3} \times n = \frac{4}{7}$

d. $\frac{11}{4} = n \times \frac{3}{4}$

e. $2\frac{1}{2} = \frac{4}{6} \times n$

f. $3\frac{2}{3} = 1\frac{1}{2} \times n$

2. The mathematical sentences in Exercise a through f are true. In which of these mathematical sentences is n a whole number? Try to find an answer without writing them on your paper.

a. $\frac{1}{2} \times n = 4$

b. $n \times \frac{2}{3} = \frac{6}{3}$

c. $n \times \frac{2}{3} = \frac{3}{3}$

d. $\frac{3}{4} \times n = 2$

e. $\frac{3}{4} \times n = 3$

f. $\frac{3}{4} \times n = 6$

Exercise Set 11

1. Find the unknown factor for each of exercises a through f. Write your answers in this form: $\frac{2}{3} \div \frac{5}{4} = \frac{8}{15}$. Use whatever other steps you need but do not write them on your paper.

   a. $\frac{3}{4} \div \frac{9}{4}$
   b. $\frac{5}{8} \div \frac{15}{8}$
   c. $\frac{1}{2} \div \frac{9}{2}$
   d. $\frac{7}{3} \div \frac{35}{3}$
   e. $\frac{11}{9} \div \frac{55}{9}$
   f. $\frac{9}{6} \div \frac{72}{6}$

2. In each of the exercises above, is the product or the known factor the greater number? Is the unknown factor less than or greater than 1?

3. Study your answers to Exercise 2. Copy and fill in the blank below so that the statement will be true. In a division example it seems that if the known factor is larger than the product, the unknown factor is ______________.

4. Follow the instructions of Exercise 1 to find the unknown factor. See if you can write the unknown factor by studying your answers to Exercise 1.

   a. $\frac{9}{4} \div \frac{3}{4}$
   b. $\frac{15}{8} \div \frac{5}{8}$
   c. $\frac{9}{2} \div \frac{1}{2}$
   d. $\frac{35}{3} \div \frac{7}{3}$
   e. $\frac{55}{9} \div \frac{11}{9}$
   f. $\frac{72}{6} \div \frac{9}{6}$

5. In each of the exercises in problem 4 is the product or the known factor the greater number? Is the unknown factor less than or greater than 1?

6. Study your answers to Exercise 5. Copy and fill in the blanks so this statement will be true.

In a division example it seems that if the known factor is less than the product, the unknown factor is ______________.

7. Write the unknown factor for each of the exercises below.

a. $\frac{32}{3} \div \frac{1}{3}$ d. $\frac{32}{3} \div \frac{8}{3}$

b. $\frac{32}{3} \div \frac{2}{3}$ e. $\frac{32}{3} \div \frac{16}{3}$

c. $\frac{32}{3} \div \frac{4}{3}$ f. $\frac{32}{3} \div \frac{32}{3}$

Study your answers, then fill in the blanks so each statement will be true.

If the known factor is multiplied by 2 and the product is unchanged, the unknown factor is ________________________.

If the known factor is divided by 2 and the product is unchanged, the unknown factor is ________________________.

Exercise Set 12

Study this set of numbers:

{1, 2, 4, 8, 16, 32, ...}

There is a rule to use in order to find a member of this set. It is: Start with 1; multiply 1 by 2 to get the next member. Any member is 2 times the member before it.

For the set {1, 4, 7, 10, 13, ...} you may think of the rule as, "Start with 1 and add 3."

Write the first 5 members of each of these sets of numbers.

1. Start with 2 and multiply by $2\frac{1}{2}$.
2. Start with $1\frac{7}{8}$ and divide by $\frac{1}{2}$.
3. Start with 288 and divide by $\frac{4}{3}$.
4. Start with 3.7 and multiply by 6.
5. Start with .19208 and multiply by .7.

6. **BRAINTWISTER**

Start with 2.1. Multiply by .4 to find the second member of the set. Add 1.2 to this second member of the set. Then repeat. The first three members of the set are: 2.1, .84, 2.04. Write the next five members.

Exercise Set 13

Tell whether each statement is always true, sometimes true, or never true.

1. The result of multiplying two rational numbers is a rational number.

2. The result of multiplying a rational number which is a whole number and a rational number less than 1 is greater than the result of multiplying two rational numbers less than 1.

3. If the product is doubled and the known factor is doubled, the unknown factor is also doubled.

4. In a division example the product is larger than the unknown factor.

5. If the known factor is larger than the product, the unknown factor is less than 1.

6. If one factor is less than 1 and the other is greater than 1, the product is less than 1.

7. If the result of multiplying two factors is greater than 1, at least one factor is greater than 1.

8. If the unknown factor is less than 1, both product and known factor are less than 1.

9. If the known factor is multiplied by 10 and the product divided by 10, the unknown factor is unchanged.

Exercise Set 14

$$4 \div 1\frac{2}{3} = 4 \div \frac{5}{3} = \frac{1}{4} \times \frac{5}{3} = \frac{5}{12}$$

WRONG

We all make silly mistakes like the one above. Often estimating answers helps us avoid silly errors. For example, for $4 \div 1\frac{2}{3}$ you could think,

$4 \div 1\frac{2}{3}$ is between $4 \div 1$ and $4 \div 2$,
so it is between 4 and 2.

You now see that $\frac{5}{12}$ is not between 4 and 2 so you have made a mistake.

Copy this chart. Fill in the blanks. Exercise 1 is done for you.

	Division	Unknown factor is between	Unknown factor is between	Unknown factor is close to
1.	$8 \div 3\frac{1}{4}$	$8 \div 4$ and $8 \div 3$	2 and 3	2
2.	$12 \div 3\frac{2}{5}$			
3.	$23\frac{7}{8} \div 6\frac{3}{4}$			
4.	$15\frac{5}{8} \div 1\frac{7}{12}$			
5.	$16\frac{5}{6} \div 6\frac{1}{2}$			
6.	$46\frac{2}{3} \div 7\frac{1}{6}$			
7.	$19\frac{4}{12} \div 4\frac{5}{8}$			

Exercise Set 15

In his science class Bob learned that a gallon of water weighs 8.36 pounds. He also found the following information about milk, turpentine and gasoline:

A gallon of milk weighs 1.03 times as much as a gallon of water.

A gallon of turpentine weighs .87 times as much as a gallon of water.

A gallon of gasoline weighs .67 times as much as a gallon of water.

Bob weighed a 10-gallon tank that he had. It weighed 18.7 pounds. A 5-gallon can weighed 10.4 pounds.

Use the information from the above paragraphs to answer these questions. Write a mathematical sentence for each question first.

1. What is the weight of a gallon of milk?
2. What will be the weight of the 10-gallon tank filled with water?
3. What is the weight of 2 gallons of gasoline?
4. What is the weight of 1 quart of water?
5. What is the weight of the 5-gallon can filled with turpentine?
6. 5 gallons of milk weighs how much more than 5 gallons of water?
7. Which weighs more, 7 gallons of gasoline or 5 gallons of turpentine?
8. Which weighs more, 5 gallons of milk and 5 gallons of gasoline, or 10 gallons of turpentine?

Exercise Set 16

Which of the mathematical sentences in the box can be used to answer each question below? Solve the mathematical sentence and write an answer sentence.

1. How many 8 inch pieces of pipe can be cut from a pipe whose length is 59.2 inches?

2. A piece of pipe 59.2 inches long is how much longer than a piece 8 inches long?

a. $8 + n = 59.2$
b. $8 \times n = 59.2$
c. $n = 59.2 \times 8$
d. $n - 8 = 59.2$
e. $8 + n = 59.2$
f. $8 - n = 59.2$
g. $n + 8 = 59.2$

3. What is the total length of eight pieces of pipe each 59.2 inches long?

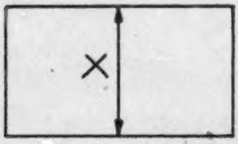
4. From what length of pipe is an 8 inch piece cut so the resulting piece is 59.2 inches long?

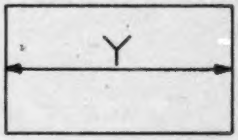
5. What is the entire length of a piece of pipe if $\frac{1}{8}$ of its length is 59.2 inches?

6. It takes 2 minutes to make each cut thru a piece of pipe. How long does it take to cut the pipe into 2 pieces?

7. Answer Exercise 6 for 3 pieces; for 4 pieces; for 5 pieces; for 10 pieces; for n pieces if n is a counting number.

8. You can cut a rectangular region into parts by vertical cuts like x or horizontal cuts like y. It takes 2 minutes to make a vertical cut and 3 minutes to make a horizontal cut. How long would it take to cut the rectangle into four equal parts if





a. You could use only vertical cuts?

b. You could use only horizontal cuts?

9. Use the information in Exercise 8 to answer these questions. You may use both horizontal and vertical cuts in answering.

a. What is the shortest time needed to cut the rectangle into four equal parts?

b. What is the shortest time needed to cut the rectangle into 8 equal parts?

c. Draw a picture to show how the cuts would be made in dividing the rectangle into 8 equal parts in 14 minutes; in 11 minutes.

Exercise Set 17

The expression $2 \times n + 1$ means, "Add 1 to the product of $2 \times n$. If $n = \frac{5}{3}$, then $2 \times n + 1$ is $(2 \times \frac{5}{3}) + 1 = \frac{10}{3} + 1 = \frac{13}{3}$."

Copy and fill in the blanks in these tables. In Exercise 1 you think, "If $n = 3$ then $2n + 1 = (2 \times 3) + 1 = 7$." Write 7 in the table below 3.

1.

If $n =$	3	8	$2\frac{1}{2}$	$5\frac{1}{3}$	$\frac{11}{8}$	0	$\frac{7}{2}$
Then $2 \times n + 1 =$							

2.

If $n =$	4	$1\frac{1}{2}$	$2\frac{1}{4}$	$5\frac{5}{6}$	$\frac{9}{7}$	$\frac{11}{3}$	8
Then $3n - 2 =$							

3.

If $n =$	2	6	3.2	4.6	.12	.78
Then $2.4n + 1.5 =$						

4.

If $n =$	3	.4	.8	100	.12	1.44
Then $(3.6 + n) + .6 =$						

5.

If $n =$	5	12	.8	.94	1.8	3.9
Then $8.48 - (n \times .6) =$						

6.

If $n =$	.96	.144	.1728	.888	.0056
Then $12.72 - (n + .08) =$					

Exercise Set 18

The square on the right is called a multiplication magic square. The product of the numbers in each row and each column is the same: for example,

$2^4 \times 2^9 \times 2^2 = 2^{15}$ $\quad$ $2^9 \times 2^5 \times 2^1 = 2^{15}$

2^4	2^3	2^8
2^9	2^5	2^1
2^2	2^7	2^6

1. Copy the magic square on the right. Fill in all empty cells so it is a multiplicative magic square.

	3^{10}	3^5
		3^4
3^7		3^9

2. You are to make a multiplicative magic square from the one at the right. Do this by changing the position of two numbers.

4^7	4^{14}	4^9
4^{13}	4^{10}	4^8
4^{11}	4^6	4^{12}

3. Is this a multiplicative magic square? Study it before you start to multiply.

16	8	256
512	32	2
4	128	64

4. Is this a multiplicative magic square?

.32	.16	5.12
10.24	.64	.04
.08	2.56	1.28

5. BRAINTWISTER. Use the number whose numerals are $5^3, 5^4, 5^5, 5^6, 5^7, 5^8, 5^9, 5^{10}, 5^{11}$ to make a multiplicative magic square. Hint: The product of the numbers in each row and column is 5^{21}.

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The following is a list of all those who have participated in the preparation of this volume.

Truman A. Botts, University of Virginia

James A. Cooley, University of Tennessee

Helen L. Curran, Glenview School, Oakland, California

Helen L. Garstens, University of Maryland

E. Glenadine Gibb, State College of Iowa

Geraldine Green, Vetal School, Detroit, Michigan

William T. Guy, Jr., University of Texas

Leon Haaland, Kenwood School, Minneapolis, Minnesota

Clarence Ethel Hardgrove, Northern Illinois University

Royce S. Hargrove, Corpus Christi Public Schools, Texas

Max Hosier, State College of Iowa

Henry G. Jacob, University of Massachusetts

Lenore John, University High School, University of Chicago

George E. Knoblock, El Carmelo School, Palo Alto, California

William G. Lister, State University of New York

Lelia M. Maneely, Springer School, Los Altos, California

John L. Marks, San Jose State College

Martha Meek, Glorietta Elementary School, Orinda, California

Mary McDermott, Mt. Diablo Unified School District, California

William K. McNabb, St. Mark's School, Dallas, Texas

Frances J. Mettler, Walter Hays Elementary School, Palo Alto, California

Leon Rutland, University of Colorado

Irene Sauble, Detroit Public Schools

Helen Schneider, LaGrange Public Schools, Illinois

Willa J. Sessions, Hillsborough County Public Schools, Florida

Rose Mary Shea, Edith C. Baker School, Brookline, Massachusetts

Wesley Thompson, Detroit Public Schools

Morgan Ward, California Institute of Technology

Ted Wassam, Ventura School, Palo Alto, California

J. Fred Weaver, Boston University